Solutions

10 points possible

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ENG EC/ME/SE 501:

Exercises (Set 1)

1pt 1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices **A,B,** and **C** of compatible dimensions that **A(BC) = (AB)C**.

(b) Prove that matrix multiplication is not commutative: i.e. it is not the case that **AB = BA** for any two square matrices **A** and **B**.

1pt 2. (a) Prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

1pt (b) Prove that if **A** is invertible, then $\mathbf{A}^{\mathrm{T}}$ is invertible.

1pt 3. (a) Suppose $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $m \times n$ and $n \times p$ matrices respectively. Find a formula for

$$\frac{d}{dt}[A(t)B(t)]$$

in terms of the derivatives of the individual matrices.

1pt (b) If $\mathbf{A}(t)$ is invertible, find a formula for the derivative (with respect to t) of its inverse.

4. Using Laplace's expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

1pt

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & x & x \\ a & ax+by & ax \\ 0 & by & 1 \end{pmatrix}$$

1pt 5. Find the inverses of the matrices in the previous problem. Assume that a, b, x, and y are integers. Under what further conditions on the symbolic entries a, b, x, y will the inverse of the second matrix have all its entries integers?

2pts 6. Let V be a finite dimensional vector space. A *basis* of V is any linearly independent set of vectors that *span* V. (a) Show that any maximal linearly independent set of vectors in V is a basis for V. (b) Show that any minimal spanning set of vectors is also a basis.

Definition: A square matrix $A = (a_{ij})$ is said to have *super-diagonal form* if $a_{ij} = 0$ for all $j < i$.

Definition: A matrix A is said to be *normal* if $AA^* = A^*A$, where A^* denotes the *Hermitian conjugate* of A (=transpose if A is real): $a^*_{ij} = \bar{a}_{ji}$, with the overbar denoting complex conjugate.

(Please turn over.)

1pt

7. Prove that any square super-diagonal matrix is normal if and only if it is diagonal.

Square matrices that are normal always have a diagonal Jordan normal form. This is a consequence of the following:

Theorem: A matrix A is unitarily similar to a diagonal matrix if and only if it is *normal*.

Exercise Set 1

$\boxed{1}$

(a) Assume the matrices of the problem, **A**, **B**, and **C**, are of the dimensions $m \times n$, $n \times p$, and $p \times q$, respectively. Allow $\mathbf{D} = \mathbf{BC}$. Then,

$$d_{ij} = \sum_{k=1}^{p} b_{ik}c_{kj}, i = 1, \ldots, n, j = 1 \ldots, q.$$

Allow $\mathbf{E} = \mathbf{AB}$. Then,

$$e_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, i = 1 \ldots, m, j = 1 \ldots, p$$

Allow $\mathbf{F} = \mathbf{AD} = \mathbf{A}(\mathbf{BC})$. Then,

$$f_{ij} = \sum_{k=1}^{n} a_{ik}d_{kj} = \sum_{k=1}^{n} a_{ik} \left\{ \sum_{h=1}^{p} b_{kh}c_{hj} \right\} = \sum_{k=1}^{n}\sum_{h=1}^{p} a_{ik}b_{kh}c_{hj}, i = 1, \ldots, m, j = 1, \ldots, q.$$

Finally, allow $\mathbf{G} = \mathbf{EC} = (\mathbf{AB})\mathbf{C}$. Then,

$$g_{ij} = \sum_{h=1}^{p} e_{ih}c_{hj} = \sum_{h=1}^{p} \left[\sum_{k=1}^{n} \{a_{ik}b_{kh}\} c_{hj} \right] = \sum_{k=1}^{n}\sum_{h=1}^{p} a_{ik}b_{kh}c_{hj}, i = 1, \ldots, m, j = 1, \ldots, q.$$

Thus, $\mathbf{F} = \mathbf{G}$, and $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

(b) Allow

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Then,

$$\mathbf{AB} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

and

$$\mathbf{BA} = \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$ and matrix multiplication is in general not commutative.

$\boxed{2}$

(a) Assuming $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, allow $\mathbf{C} = \mathbf{AB}$. Then,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, i, j = 1, \ldots, n.$$

Allow $\mathbf{D} = \mathbf{C}^T$. Then,

$$d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}, i, j = 1, \ldots, n.$$

Allow $\mathbf{E} = \mathbf{B}^T \mathbf{A}^T$. Then,

$$e_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk}, i, j = 1, \ldots, n.$$

Thus, $\mathbf{D} = \mathbf{E}$ and therefore $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

(b) If the matrix $\mathbf{A}$ is invertible, there exist a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I}.$$

Taking the transpose of both sides of this equation gives

$$\mathbf{B}^T \mathbf{A}^T = \mathbf{I}^T = \mathbf{I}$$

and hence, $\mathbf{A}^T$ has an inverse.

$\boxed{3}$

(a) Assigning $\mathbf{C}(t) = \mathbf{A}(t)\mathbf{B}(t)$, yields

$$c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t) b_{kj}(t).$$

This implies that

$$\frac{d}{dt}\left(c_{ij}(t)\right) = \sum_{k=1}^{n} \left[\frac{d}{dt}\left(a_{ik}(t)\right) b_{kj}(t) + a_{ik}(t) \frac{d}{dt}\left(b_{kj}(t)\right) \right].$$

Simplification of the above yields

$$\frac{d}{dt}\left(c_{ij}(t)\right) = \sum_{k=1}^{n} \frac{d}{dt}\left(a_{ik}(t)\right) b_{kj}(t) + \sum_{k=1}^{n} a_{ik}(t) \frac{d}{dt}\left(b_{kj}(t)\right).$$

From the above it is clear that

$$\frac{d}{dt}\left(\mathbf{C}(t)\right) = \frac{d}{dt}\left(\mathbf{A}(t)\mathbf{B}(t)\right) = \frac{d}{dt}\left(\mathbf{A}(t)\right)\mathbf{B}(t) + \mathbf{A}(t)\frac{d}{dt}\left(\mathbf{B}(t)\right).$$

(b) If $\mathbf{A}(t)$ is invertible, there exist a matrix $\mathbf{B}(t)$ such that

$$\mathbf{A}(t)\mathbf{B}(t) = \mathbf{I}.$$

Taking the derivative of both sides gives

$$\frac{d}{dt}\left(\mathbf{A}(t)\mathbf{B}(t)\right) = \frac{d}{dt}\left(\mathbf{I}\right) = \mathbf{0},$$

where $\mathbf{0}$ is the matrix of zeros.

From part (a), the expression becomes

$$\frac{d}{dt}\left(\mathbf{A}(t)\right)\mathbf{B}(t) + \mathbf{A}(t)\frac{d}{dt}\left(\mathbf{B}(t)\right) = \mathbf{0}.$$

Moving the first term to the right hand side of the equation gives

$$\mathbf{A}(t)\frac{d}{dt}\left(\mathbf{B}(t)\right) = -\frac{d}{dt}(\mathbf{A}(t))\mathbf{B}(t).$$

After left-multiplying both sides of the equation by $\mathbf{B}(t)$, the expression for the derivative of the inverse of $\mathbf{A}(t)$ with respect to t becomes

$$\frac{d}{dt}(\mathbf{B}(t)) = -\mathbf{B}(t)\frac{d}{dt}(\mathbf{A}(t))\mathbf{B}(t).$$

$\boxed{4}$

(a)

$$\begin{vmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$
$$= 3(8-3) + (2-4) = 3(5) - 2 = 13.$$

(b)

$$1\begin{vmatrix} ax+by & ax \\ by & 1 \end{vmatrix} - a\begin{vmatrix} x & x \\ by & 1 \end{vmatrix} = by$$

$\boxed{5}$

(a)

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 - 3R_1 \to R_3}{\xrightarrow{R_2 - 2R_1 \to R_2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & -2 \\ 0 & -3 & -5 & 1 & 0 & -3 \end{array}\right] \xrightarrow{\frac{R_2}{2} \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & -3 & -5 & 1 & 0 & -3 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \to R_1 \\ \longrightarrow \\ R_3 + 3R_2 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{2} & 0 & \frac{-1}{2} & 2 \\ 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{-13}{2} & 1 & \frac{3}{2} & -6 \end{array}\right]$$

$$\begin{array}{c} \frac{-2R_3}{13} \to R_3 \\ \longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{2} & 0 & \frac{-1}{2} & 2 \\ 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{-2}{13} & \frac{-3}{13} & \frac{12}{13} \end{array}\right]$$

$$\begin{array}{c} R_1 - \frac{5R_3}{2} \to R_1 \\ \longrightarrow \\ R_2 + \frac{R_3}{2} \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{13} & \frac{1}{13} & \frac{-4}{13} \\ 0 & 1 & 0 & \frac{-1}{13} & \frac{5}{13} & \frac{-7}{13} \\ 0 & 0 & 1 & \frac{-2}{13} & \frac{-3}{13} & \frac{12}{13} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \frac{1}{13}\begin{bmatrix} 5 & 1 & -4 \\ -1 & 5 & -7 \\ -2 & -3 & 12 \end{bmatrix}.$$

(b) Since the determinant is by, the determinant of the inverse is $\frac{1}{by}$. Given that b and y are assumed to be integers, this will be an integer if and only if b and y are both equal to 1 or -1.

$\boxed{6}$

(a) Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ be a maximal linearly independent set of vectors in V, and let $\mathbf{v} in V$ be an arbitrarily chosen vector. The set $\{\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{v}\}$ must be *linearly dependent*. This means that there is a set of scalars $\{a_1, \ldots, a_{n+1}\}$, not all of which are zero, such that

$$a_1\mathbf{p}_1 + \cdots + a_n\mathbf{p}_n + a_{n+1}\mathbf{v} = 0.$$

In this expression, it must be the case that $a_{n+1} \neq 0$, becasue otherwise there would be a nontrivial linear combination of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ that is equal to zero. This would contradict the linear independence of $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$.

Hence we may write

$$\mathbf{v} = -\frac{a_1}{a_{n+1}}\mathbf{p}_1 - \cdots - \frac{a_n}{a_{n+1}}\mathbf{p}_n.$$

Since $\mathbf{v}$ was arbitrary, this proves that $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ spans V.

(b) Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ be a minimal spanning set, and suppose there is a nontrivial linear combination

$$a_1\mathbf{p}_1 + \cdots + a_n\mathbf{p}_n = 0.$$

We may assume without loss of generality that $a_n \neq 0$. Then we can write

$$\mathbf{p}_n = -\frac{a_1}{a_n}\mathbf{p}_1 - \cdots - \frac{a_{n-1}}{a_n}\mathbf{p}_{n-1}.$$

Let $\mathbf{v} in V$ be an arbitrarily chosen vector. Since $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is a spanning set, there are scalars $\beta_1 \ldots, \beta_n$ such that

$$\mathbf{v} = \beta_1\mathbf{p}_1 + \cdots + \beta_n\mathbf{p}_n.$$

But we can also write

$$\mathbf{v} = (\beta_1 - \frac{a_1\beta_n}{a_n})\mathbf{p}_1 + \cdots + (\beta_{n-1} - \frac{a_{n-1}\beta_n}{a_n}\mathbf{p}_{n-1}.$$

Since $\mathbf{v}$ was chosen arbitrarily, this means that any vector in V may be expressed as a linear combination of $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$ contradicting the assumption that $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is a minimal spanning set.

$\boxed{7}$

A matrix A is *normal* if and only if

$$AA^* = A^*A \qquad (1)$$

($AA^T = A^TA$ in the case A is real). Compare diagonal entries in the matrix products on both sides of equation (1).

$$\sum_{k=1}^{n} |a_{ik}|^2 = \sum_{k=1}^{n} |a_{ki}|^2, \qquad (i = 1, \ldots, n).$$

Because A is super-diagonal, this equation can be rewritten

$$\sum_{k=i}^{n} |a_{ik}|^2 = \sum_{k=1}^{i} |a_{ki}|^2, \qquad (i = 1, \ldots, n).$$

Let i be the smallest integer such that $a_{ij} \neq 0$ for some j with $j > 0$. Then (1) implies for this value of i that

$$\sum_{k=i}^{n} |a_{ik}|^2 = |a_{ii}|^2.$$

Clearly this equation cannot hold if $a_{ij} \neq 0$ for any $j > 0$. This contradiction proves that:

Any super-diagonal matrix is normal if and only if it is diagonal.

$\boxed{\text{Additional}}$

If

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -3 & 2 & -5 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 3 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 & -3 & 2 & -5 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 1 & 0 & 1-\lambda & 1 & -2 \\ 0 & -1 & 0 & 3-\lambda & 1 \\ 1 & -1 & -1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 5-\lambda & -3 & 2 & -5 \\ 1 & 1-\lambda & 1 & -2 \\ 0 & 0 & 3-\lambda & 1 \\ 1 & -1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \left\{ (3-\lambda) \begin{vmatrix} 5-\lambda & -3 & -5 \\ 1 & 1-\lambda & -2 \\ 1 & -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 5-\lambda & -3 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & -1 & 1 \end{vmatrix} \right\}$$

$$= (2-\lambda)\{(3-\lambda)[(5-\lambda)((1-\lambda)^2 - 2) + 3(3-\lambda) + 5(2-\lambda)]$$

$$
\begin{aligned}
&-(5-\lambda)(2-\lambda)-2(2-\lambda)\}\\
&=(2-\lambda)[\lambda^4-10\lambda^3+37\lambda^2-60\lambda+36]\\
&=(2-\lambda)[(2-\lambda)^2(3-\lambda)^2]=(2-\lambda)^3(3-\lambda)^2.
\end{aligned}
$$