Solutions

10 points possible

ENG EC/ME/SE 51:

Exercises (Set 1)

1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices $A$, $B$, and $C$ of compatible dimensions that $A(BC) = (AB)C$.

(b) Prove that matrix multiplication is not commutative: i.e. it is not the case that $AB = BA$ for any two square matrices $A$ and $B$.

2. (a) Prove that $(AB)^T = B^T A^T$.

(b) Prove that if $A$ is invertible, then $A^T$ is invertible.

3. (a) Suppose $A(t)$ and $B(t)$ are $m \times n$ and $n \times p$ matrices respectively. Find a formula for $\frac{d}{dt}[A(t)B(t)]$ in terms of the derivatives of the individual matrices.

(b) If $A(t)$ is invertible, find a formula for the derivative (with respect to $t$) of its inverse.

4. Using Laplace’s expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

\[
\begin{pmatrix}
3 & 0 & 1 \\
2 & 4 & 3 \\
1 & 1 & 2
\end{pmatrix}
\quad
\begin{pmatrix}
1 & x & x \\
a & ax + by & ax \\
0 & by & 1
\end{pmatrix}
\]

5. Find the inverses of the matrices in the previous problem. Assume that $a, b, x, y$ are integers. Under what further conditions on the symbolic entries $a, b, x, y$ will the inverse of the second matrix have all its entries integers?

6. Let $V$ be a finite dimensional vector space. A basis of $V$ is any linearly independent set of vectors that span $V$. (a) Show that any maximal linearly independent set of vectors in $V$ is a basis for $V$. (b) Show that any minimal spanning set of vectors is also a basis.

Definition: A square matrix $A = (a_{ij})$ is said to have super-diagonal form if $a_{ij} = 0$ for all $j < i$.

Definition: A matrix $A$ is said to be normal if $AA^* = A^*A$, where $A^*$ denotes the Hermitian conjugate of $A$ (=transpose if $A$ is real): $a_{ij}^* = \overline{a_{ji}}$, with the overbar denoting complex conjugate.

(Please turn over.)
7. Prove that any square super-diagonal matrix is normal if and only if it is diagonal.

Square matrices that are normal always have a diagonal Jordan normal form. This is a consequence of the following:

**Theorem:** A matrix $A$ is unitarily similar to a diagonal matrix if and only if it is normal.
(a) Assume the matrices of the problem, \( A, B, \) and \( C, \) are of the dimensions \( m \times n, n \times p, \) and \( p \times q, \) respectively. Allow \( D = BC. \) Then,

\[
d_{ij} = \sum_{k=1}^{p} b_{ik} c_{kj}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, q.
\]

Allow \( E = AB. \) Then,

\[
e_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, p.
\]

Allow \( F = AD = A(BC). \) Then,

\[
f_{ij} = \sum_{k=1}^{n} a_{ik} d_{kj} = \sum_{k=1}^{n} a_{ik} \left\{ \sum_{h=1}^{p} b_{kh} c_{hj} \right\} = \sum_{k=1}^{n} \sum_{h=1}^{p} a_{ik} b_{kh} c_{hj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, q.
\]

Finally, allow \( G = EC = (AB)C. \) Then,

\[
g_{ij} = \sum_{h=1}^{p} e_{ih} c_{hj} = \sum_{h=1}^{p} \left[ \sum_{k=1}^{n} \{ a_{ik} b_{kh} \} \right] c_{hj} = \sum_{k=1}^{n} \sum_{h=1}^{p} a_{ik} b_{kh} c_{hj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, q.
\]

Thus, \( F = G, \) and \( A(BC) = (AB)C. \)

(b) Allow

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}
\]

Then,

\[
AB = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}
\]

and

\[
BA = \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}
\]

Hence, \( AB \neq BA \) and matrix multiplication is in general not commutative.
(a) Assuming \( A \) and \( B \) are \( n \times n \) matrices, allow \( C = AB \). Then,

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i, j = 1, \ldots, n.
\]

Allow \( D = C^T \). Then,

\[
d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}, \quad i, j = 1, \ldots, n.
\]

Allow \( E = B^T A^T \). Then,

\[
e_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk}, \quad i, j = 1, \ldots, n.
\]

Thus, \( D = E \) and therefore \((AB)^T = B^T A^T\).

(b) If the matrix \( A \) is invertible, there exist a matrix \( B \) such that

\[
AB = I.
\]

Taking the transpose of both sides of this equation gives

\[
B^T A^T = I^T = I
\]

and hence, \( A^T \) has an inverse.

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(a) Assigning \( C(t) = A(t)B(t) \), yields

\[
c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t) b_{kj}(t).
\]

This implies that

\[
\frac{d}{dt}(c_{ij}(t)) = \sum_{k=1}^{n} \left[ \frac{d}{dt}(a_{ik}(t)) b_{kj}(t) + a_{ik}(t) \frac{d}{dt}(b_{kj}(t)) \right].
\]

Simplification of the above yields

\[
\frac{d}{dt}(c_{ij}(t)) = \sum_{k=1}^{n} \frac{d}{dt}(a_{ik}(t)) b_{kj}(t) + \sum_{k=1}^{n} a_{ik}(t) \frac{d}{dt}(b_{kj}(t)).
\]

From the above it is clear that

\[
\frac{d}{dt}(C(t)) = \frac{d}{dt}(A(t)B(t)) = \frac{d}{dt}(A(t)) B(t) + A(t) \frac{d}{dt}(B(t)).
\]
(b) If \( A(t) \) is invertible, there exist a matrix \( B(t) \) such that

\[
A(t)B(t) = I.
\]

Taking the derivative of both sides gives

\[
\frac{d}{dt} (A(t)B(t)) = \frac{d}{dt} (I) = 0,
\]

where \( 0 \) is the matrix of zeros.

From part (a), the expression becomes

\[
\frac{d}{dt} (A(t)) B(t) + A(t) \frac{d}{dt} (B(t)) = 0.
\]

Moving the first term to the right hand side of the equation gives

\[
A(t) \frac{d}{dt} (B(t)) = - \frac{d}{dt} (A(t)) B(t).
\]

After left-multiplying both sides of the equation by \( B(t) \), the expression for the derivative of the inverse of \( A(t) \) with respect to \( t \) becomes

\[
\frac{d}{dt} (B(t)) = -B(t) \frac{d}{dt} (A(t)) B(t).
\]

\[
\begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

\[
= 3(8 - 3) + (2 - 4) = 3(5) - 2 = 13.
\]

(b)

\[
\begin{array}{c|ccc}
1 & ax + by & ax \\ ax & by & 1 - a & x \\ x & by & 1 & = by
\end{array}
\]

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(a)

\[
\begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} R_1 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 -3 & -5 \end{bmatrix} R_2 - 2R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix} R_3 - 3R_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix} R_2 \rightarrow R_2 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix} R_3 \rightarrow R_3 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 -3 & -5 \end{bmatrix}
\]
\[
\begin{align*}
R_1 - R_2 & \to R_1 & \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & - \frac{1}{2} \\ 0 & 0 & - \frac{3}{2} \end{bmatrix} & \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ 1 & \frac{3}{2} & -1 \end{bmatrix} \\
R_3 + 3R_2 & \to R_3 & \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & - \frac{1}{2} \\ 0 & 0 & - \frac{3}{2} \end{bmatrix} & \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ 1 & \frac{3}{2} & -1 \end{bmatrix} \\
-\frac{2R_3}{13} & \to R_3 & \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & - \frac{1}{2} \\ 0 & 0 & - \frac{3}{2} \end{bmatrix} & \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ 1 & \frac{3}{2} & -1 \end{bmatrix} \\
R_1 - \frac{5R_3}{2} & \to R_1 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \frac{5}{13} & \frac{1}{13} & \frac{1}{13} \\ \frac{13}{13} & \frac{13}{13} & \frac{13}{13} \end{bmatrix} \\
R_2 + \frac{R_3}{2} & \to R_2 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \frac{5}{13} & \frac{1}{13} & \frac{1}{13} \\ \frac{13}{13} & \frac{13}{13} & \frac{13}{13} \end{bmatrix} \\
\Rightarrow & \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}^{-1} & = \frac{1}{13} & \begin{bmatrix} 5 & 1 & -4 \\ -1 & 5 & -7 \\ -2 & -3 & 12 \end{bmatrix}.
\end{align*}
\]

(b) Since the determinant is by, the determinant of the inverse is \(1/\det \). Given that \(b\) and \(y\) are assumed to be integers, this will be an integer if and only if \(b\) and \(y\) are both equal to 1 or -1.

6

(a) Let \(\{p_1, \ldots, p_n\}\) be a maximal linearly independent set of vectors in \(V\), and let \(\text{vin} V\) be an arbitrarily chosen vector. The set \(\{p_1, \ldots, p_n, v\}\) must be linearly dependent. This means that there is a set of scalars \(\{a_1, \ldots, a_{n+1}\}\), not all of which are zero, such that

\[
a_1p_1 + \cdots + a_n p_n + a_{n+1}v = 0.
\]

In this expression, it must be the case that \(a_{n+1} \neq 0\), because otherwise there would be a nontrivial linear combination of \(p_1, \ldots, p_n\) that is equal to zero. This would contradict the linear independence of \(\{p_1, \ldots, p_n\}\).

Hence we may write

\[
v = -\frac{a_1}{a_{n+1}}p_1 - \cdots - \frac{a_n}{a_{n+1}}p_n.
\]

Since \(v\) was arbitrary, this proves that \(\{p_1, \ldots, p_n\}\) spans \(V\).

(b) Let \(\{p_1, \ldots, p_n\}\) be a minimal spanning set, and suppose there is a nontrivial linear combination

\[
a_1p_1 + \cdots + a_n p_n = 0.
\]

We may assume without loss of generality that \(a_n \neq 0\). Then we can write

\[
p_n = -\frac{a_1}{a_n}p_1 - \cdots - \frac{a_{n-1}}{a_n}p_{n-1}.
\]

Let \(\text{vin} V\) be an arbitrarily chosen vector. Since \(\{p_1, \ldots, p_n\}\) is a spanning set, there are scalars \(\beta_1, \ldots, \beta_n\) such that

\[
v = \beta_1p_1 + \cdots + \beta_n p_n.
\]

But we can also write

\[
v = (\beta_1 - \frac{a_1}{a_n} \beta_n)p_1 + \cdots + (\beta_{n-1} - \frac{a_{n-1}}{a_n} \beta_n)p_{n-1}.
\]
Since $v$ was chosen arbitrarily, this means that any vector in $V$ may be expressed as a linear combination of $p_1, \ldots, p_{n-1}$ contradicting the assumption that $\{p_1, \ldots, p_n\}$ is a minimal spanning set.

7

A matrix $A$ is normal if and only if

$$AA^* = A^*A$$  \hspace{1cm} (1)

($AA^T = A^TA$ in the case $A$ is real). Compare diagonal entries in the matrix products on both sides of equation (1).

$$\sum_{k=1}^{n} |a_{ik}|^2 = \sum_{k=1}^{n} |a_{ki}|^2, \quad (i = 1, \ldots, n).$$

Because $A$ is super-diagonal, this equation can be rewritten

$$\sum_{k=i}^{n} |a_{ik}|^2 = \sum_{k=1}^{i} |a_{ki}|^2, \quad (i = 1, \ldots, n).$$

Let $i$ be the smallest integer such that $a_{ij} \neq 0$ for some $j$ with $j > 0$. Then (1) implies for this value of $i$ that

$$\sum_{k=i}^{n} |a_{ik}|^2 = |a_{ii}|^2.$$

Clearly this equation cannot hold if $a_{ij} \neq 0$ for any $j > 0$. This contradiction proves that:

Any super-diagonal matrix is normal if and only if it is diagonal.

**Additional**

If

$$A = \begin{bmatrix}
5 & -1 & -3 & 2 & -5 \\
0 & 2 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -2 \\
0 & -1 & 0 & 3 & 1 \\
1 & -1 & -1 & 1 & 1
\end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix}
5 - \lambda & -1 & -3 & 2 & -5 \\
0 & 2 - \lambda & 0 & 0 & 0 \\
1 & 0 & 1 - \lambda & 1 & -2 \\
0 & -1 & 0 & 3 - \lambda & 1 \\
1 & -1 & -1 & 1 & 1 - \lambda
\end{vmatrix} = (2 - \lambda) \begin{vmatrix}
5 - \lambda & -3 & 2 & -5 \\
0 & 0 & 3 - \lambda & 1 \\
1 & -1 & 1 & 1 - \lambda
\end{vmatrix}$$

$$= (2 - \lambda) \left\{ (3 - \lambda) \begin{vmatrix}
5 - \lambda & -3 & -5 \\
1 & 1 - \lambda & 1 - \lambda \\
1 & -1 & 1 - \lambda
\end{vmatrix} - \begin{vmatrix}
5 - \lambda & -3 & 2 \\
1 & 1 - \lambda & 1 \\
1 & -1 & 1 - \lambda
\end{vmatrix} \right\}$$

$$= (2 - \lambda) \left\{ (3 - \lambda)((5 - \lambda)((1 - \lambda)^2 - 2) + 3(3 - \lambda) + 5(2 - \lambda)) \right\}$$
\[-(5 - \lambda)(2 - \lambda) - 2(2 - \lambda)\}
= (2 - \lambda)[\lambda^4 - 10\lambda^3 + 37\lambda^2 - 60\lambda + 36]
= (2 - \lambda)[(2 - \lambda)^2(3 - \lambda)^2] = (2 - \lambda)^3(3 - \lambda)^2.