ENG EC/ME/SE 501:

**Exercises (Set 1)** (Due 9/17/20)

1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices $A, B,$ and $C$ of compatible dimensions that $A(BC) = (AB)C$.

(b) Prove that matrix multiplication is not commutative: i.e. it is not the case that $AB = BA$ for any two square matrices $A$ and $B$.

2. (a) Prove that $(AB)^T = B^TA^T$.

(b) Prove that if $A$ is invertible, then $A^T$ is invertible.

3. (a) Suppose $A(t)$ and $B(t)$ are $m \times n$ and $n \times p$ matrices respectively. Find a formula for

$$\frac{d}{dt}[A(t)B(t)]$$

in terms of the derivatives of the individual matrices.

(b) If $A(t)$ is invertible, find a formula for the derivative (with respect to $t$) of its inverse.

4. Using Laplace’s expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & x & x \\ a & ax + by & ax \\ 0 & by & 1 \end{pmatrix}$$

5. Find the inverses of the matrices in the previous problem. Assume that $a, b, x, y$ are integers. Under what further conditions on the symbolic entries $a, b, x, y$ will the inverse of the second matrix have all its entries integers?

6. Let $V$ be a finite dimensional vector space. A basis of $V$ is any linearly independent set of vectors that span $V$. (a) Show that any maximal linearly independent set of vectors in $V$ is a basis for $V$. (b) Show that any minimal spanning set of vectors is also a basis.

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**Definition:** Given a square matrix $A$, the determinant of the $(n-1) \times (n-1)$ submatrix $M_{ij}$ obtained from $A$ by deleting the $i$-th row and $j$-th column is called the $ij$-th minor of $A$. Multiplying the $ij$-minor by $-1^{i+j}$ yields the $ij$-th cofactor $A_{ij}$. Among the list of determinant facts that were recalled in class, we have the following:

$$\det A = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij}.$$
7. The adjugate of $A$ is the $n \times n$ matrix adj$(A)$ s.t. adj$(A)_{ij} = A_{ji}$. Using the properties of determinants reviewed in class, prove the following.

**Theorem:** If $\det A \neq 0$, then the matrix $A$ is invertible, and when this is the case

$$A^{-1} = \left(1/\det A\right) \text{adj}(A).$$