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All homework must be uploaded to the Blackboard site no later than noon Boston time on the posted due date. PDF is the preferred format.

ENG EC/ME/SE 501:

Exercises (Set 1) (Due 9/19/24)

1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices **A,B,** and **C** of compatible dimensions that **A(BC) = (AB)C**.

(b) Prove that matrix multiplication is <u>not</u> commutative: i.e. it is not the case that **AB = BA** for any two square matrices **A** and **B**.

2. (a) Prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

(b) Prove that if **A** is invertible, then $\mathbf{A}^{\mathrm{T}}$ is invertible.

3. (a) Suppose $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $m \times n$ and $n \times p$ matrices respectively. Find a formula for

$$\frac{d}{dt}[A(t)B(t)]$$

in terms of the derivatives of the individual matrices.

(b) If $\mathbf{A}(t)$ is invertible, find a formula for the derivative (with respect to t) of its inverse.

4. Using Laplace's expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & x & x \\ a & ax+by & ax \\ 0 & by & 1 \end{pmatrix}$$

5. Find the inverses of the matrices in the previous problem. Assume that a, b, x, and y are integers. Under what further conditions on the symbolic entries a, b, x, y will the inverse of the second matrix have all its entries integers?

6. Let V be a finite dimensional vector space. A *basis* of V is any linearly independent set of vectors that *span* V. (a) Show that any maximal linearly independent set of vectors in V is a basis for V. (b) Show that any minimal spanning set of vectors is also a basis.

(Please turn over.)

Definition: Given a square matrix A, the determinant of the $(n-1)\times(n-1)$ submatrix M_{ij} obtained from A by deleting the i-th row and j-th column is called the ij-j *minor* of A. Multiplying the ij-minor by -1^{i+j} yields the ij-th *cofactor* A_{ij}. Among the list of determinant facts that were recalled in class, we have the following:

$$\det \mathsf{A} = \sum_{i=1}^{n} a_{ij}\mathsf{A}_{ij} = \sum_{j=1}^{n} a_{ij}\mathsf{A}_{ij}.$$

7. The *adjugate* of A is the $n \times n$ matrix adj**(A)** s.t. $\mathrm{adj}(\mathsf{A})_{ij} = \mathsf{A}_{ji}$. Using the properties of determinants reviewed in class, prove the following.

Theorem: If $\det \mathsf{A} \neq 0$, then the matrix A is invertible, and when this is the case

$$\mathsf{A}^{-1} = (1/\det \mathsf{A})\ \mathrm{adj}(\mathsf{A}).$$