ENG EC/ME/SE 501:

Exercises (Set 1)  (Due 9/20/17)

1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices $A, B,$ and $C$ of compatible dimensions that $A(BC) = (AB)C$.

(b) Prove that matrix multiplication is not commutative: i.e. it is not the case that $AB = BA$ for any two square matrices $A$ and $B$.

2. (a) Prove that $(AB)^T = B^T A^T$.

(b) Prove that if $A$ is invertible, then $A^T$ is invertible.

3. (a) Suppose $A(t)$ and $B(t)$ are $m \times n$ and $n \times p$ matrices respectively. Find a formula for $\frac{d}{dt}[A(t)B(t)]$ in terms of the derivatives of the individual matrices.

(b) If $A(t)$ is invertible, find a formula for the derivative (with respect to $t$) of its inverse.

4. Using Laplace’s expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{pmatrix}$  $\begin{pmatrix} 1 & x & x \\ a & ax + by & ax \\ 0 & by & 1 \end{pmatrix}$

5. Find the inverses of the matrices in the previous problem. Assume that $a, b, x, y$ are integers. Under what further conditions on the symbolic entries $a, b, x, y$ will the inverse of the second matrix have all its entries integers?

6. Let $V$ be a finite dimensional vector space. A basis of $V$ is any linearly independent set of vectors that span $V$. (a) Show that any maximal linearly independent set of vectors in $V$ is a basis for $V$. (b) Show that any minimal spanning set of vectors is also a basis.

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**Definition:** A square matrix $A = (a_{ij})$ is said to have super-diagonal form if $a_{ij} = 0$ for all $j < i$.

**Definition:** A matrix $A$ is said to be normal if $AA^* = A^*A$, where $A^*$ denotes the Hermitian conjugate of $A$ (=transpose if $A$ is real): $a_{ij}^* = \overline{a_{ji}}$, with the overbar denoting complex conjugate.

(Please turn over.)
7. Prove that any square super-diagonal matrix is normal if and only if it is diagonal.

Square matrices that are normal always have a diagonal Jordan normal form. This is a consequence of the following:

**Theorem:** A matrix $A$ is unitarily similar to a diagonal matrix if and only if it is *normal*. 
