ME/SC 501 SOLUTIONS - HOMEWORK 8 PROF. BAILLIEUL

1. WRITE THIS AS A FIRST-ORDER LINEAR SYSTEM

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{w} .$$

THIS IS OF THE FORM

$$\dot{y} = Ay + b\dot{w} .$$

FORMALLY APPLYING THE VARIATION OF CONSTANTS FORMULA

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} b \dot{w}(s) ds .$$

TAKING EXPECTATIONS OF BOTH SIDES, THE INTEGRAL TERM VANISHES AND WE HAVE

$$\bar{y}(t) = e^{At} y_0 .$$

SINCE BOTH EIGENVALUES OF A ARE IN THE O.L.H.P., $\bar{y}(t) \longrightarrow 0$ AS $t \to \infty$, AND THE

STEADY-STATE MEAN IS ZERO. TO OBTAIN THE STEADY-STATE VARIANCE, RECALL FROM THE LECTURE THAT THE COVARIANCE MATRIX $\Sigma(t)$ SATISFIES

$$\dot{\Sigma}(t) = \Sigma A^T + A\Sigma + bQb^T$$

IN OUR CASE $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Q = 1$ (A SCALAR)

SO THAT

$$bQb^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

THE STEADY STATE SOLUTION MAY BE FOUND BY SOLVING

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \Sigma A^T + A\Sigma + bQb^T$$

$$= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

WRITING THE COMPONENT EQUATIONS

$$2\sigma_{12} = 0$$

$$-\sigma_{11} - \sigma_{12} + \sigma_{22} = 0$$

$$-2\sigma_{12} - 2\sigma_{22} + 1 = 0$$

THE SOLUTION IS $\sigma_{11} = 1/2$, $\sigma_{12} = 0$, $\sigma_{22} = 1/2$.

Hence

$$\lim_{t \to \infty} E\left[x(t)^2\right] = \sigma_{11} = 1/2 .$$

THE STEADY STATE PDF IS

$$\frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_{11}}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-x^2} .$$

2. $\dot{\hat{x}} = \left(A - \Sigma^\circ C^T R^{-1} C\right)\hat{x} + \Sigma^\circ C^T R^{-1} y$

$\hat{x}(0) = E(x_0)$

$$\dot{\Sigma}^\circ = A\Sigma^\circ + \Sigma^\circ A^T + FQF^T - \Sigma^\circ C^T R^{-1} C \Sigma^\circ$$

$$\Sigma^\circ(0) = E\left[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right]$$

$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad C = (1, 0)$ $\quad Q = q,\ R = 1$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} - \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}}_{\text{STEADY STATE COVARIANCE EQU.}}$$