Solutions

ENG EC/ME/SC 501:

Exercises (Set 7)  (Due 12/6/18)

1. Given that \( x(0) = 1 \), find \( x(\cdot) \) on the interval \( 0 \leq t \leq T \) such that

\[
J = \int_0^T \dot{x}^2 + x^2 \, dt
\]

is minimized.  (*Hint:* Convert this into a control problem by setting \( \dot{x} = u \).)

2. Suppose that the partitioned system

\[
\begin{pmatrix}
\dot{w}(t) \\
\dot{y}(t)
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
w(t) \\
y(t)
\end{pmatrix}
\]

with output \( y(t) \) is observable. Show that \( \{A_{11}, A_{21}\} \) is an observable pair.

3. (a) Consider the linear system

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix}
+ \begin{pmatrix}
0 \\
u(t)
\end{pmatrix}.
\]

(1)

For \( T > 0 \), find the control input that steers the state of (1) from \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) to \( \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \) in \( T \) units of time so as to minimize the performance metric

\[
\eta = \int_0^T u(t)^2 \, dt.
\]

(2)

(b) For \( \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \), evaluate \( \eta \). Why are the values different?
Exercise Set 7
Solutions

1

Allow $\dot{x} = u$. Note that this represents a controllable system. Then

$$J = \int_0^T x^2 + u^2 dt = \int_0^T u^2 + x^2 dt.$$  

This is equivalent to

$$J = \int_0^T x^T Q x + u^T R u dt,$$

where $Q = R = 1$. Note that both $Q$ and $R$ are positive definite.

The associated Ricatti equation is

$$\dot{m} = (m + 1)(m - 1),$$  \hspace{1cm} (1)

which is separable, simplifying to the equation of integrals

$$\int \frac{dm}{(m + 1)(m - 1)} = \int dt.$$  \hspace{1cm} (2)

The integral on the right hand side of (2) produces $t + c$. The integral on the left hand side of (2), after consulting an integration table, evaluates to

$$\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right).$$

Thus, equation (2) becomes

$$\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right) = t + c,$$

which, after some algebra, produces the solution to (1) as

$$m(t) = \frac{1 + ce^{2t}}{1 - ce^{2t}}.$$  \hspace{1cm} (3)

In order for $m(\cdot)$ to pass through 0 at time $t = T$, $c$ must be $-e^{-2T}$. Accordingly, the solution in (3) becomes

$$m(t) = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T)}} = -\tanh(t - T).$$  \hspace{1cm} (4)

Since $m(t) \geq 0$ for all $t \in [0, T]$, $m(t)$ is positive semidefinite on the interval $[0, T]$. Therefore, the control that minimizes $J$ is given by

$$u(t) = x(t) \tanh(t - T).$$  \hspace{1cm} (5)

Thus $x(\cdot)$ satisfies

$$\dot{x}(t) = x(t) \tanh(t - T).$$

Solving for $x(\cdot)$,

$$\frac{dx}{dt} = x \tanh(t - T),$$

$$\int \frac{dx}{x} = \int \tanh(t - T) dt$$

$$\ln x = \ln \cosh(t - T) + c$$

$$x(t) = c \cosh(t - T).$$
Employing the initial condition produces

\[ x(0) = 1 = c \cosh(-T) = c \cosh(T) \]
\[ e = \text{sech}(T). \]

And thus,

\[ x(t) = \text{sech}(T) \cosh(t - T). \]

We have

\[ \dot{w} = A_{11}w + A_{12}y \]
\[ \dot{y} = A_{21}w + A_{22}y. \]

Allow \( z = \dot{y} - A_{22}y \). Since \( y \) is observable, \( z \) is a known quantity.

The solution for \( w \) is obtained by

\[ w(t) = e^{A_{11}t}w_0 + f(y) \]

where \( f \) is some function of observable \( y \) and therefore evaluates to a known quantity. This gives

\[ z(t) = A_{21}e^{A_{11}t}w_0 + f'(y), \]

where \( f' \) is some other function of observable \( y \) and therefore evaluates to a known quantity. Differentiating \( z \) produces

\[ \dot{z}(t) = A_{21}A_{11}e^{A_{11}t}w_0 + f''(y), \]

where again \( f'' \) is some function that evaluates to a known quantity. Eventually, after repeated differentiation and evaluation, we obtain

\[
\begin{bmatrix}
  z(0) \\
  \dot{z}(0) \\
  \vdots \\
  z^{(n)}(0)
\end{bmatrix}
= 
\begin{bmatrix}
  A_{21} \\
  A_{21}A_{11} \\
  \vdots \\
  A_{21}A_{11}^{n-1}
\end{bmatrix}
\begin{bmatrix}
  w_0 \\
  F(y)
\end{bmatrix},
\]

where \( F \) is some vector that we can assume is known. Since \( F, w_0, \) and \([z(0), \dot{z}(0), \ldots, z^{(n)}(0)]^T\) are all known quantities, the matrix

\[
\begin{bmatrix}
  A_{21} \\
  A_{21}A_{11} \\
  \vdots \\
  A_{21}A_{11}^{n-1}
\end{bmatrix}
\]

is invertible, and therefore full rank. Thus, \((A_{11}, A_{21})\) is an observable pair.
3(a).

\[ W(0, T) = \int_0^T \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} ds \]

\[ = \begin{pmatrix} \frac{T^3}{3} & -\frac{T^2}{2} \\ -\frac{T^2}{2} & T \end{pmatrix} \]

\[ W(0, T)^{-1} = \begin{pmatrix} \frac{12}{T^3} & \frac{6}{T^2} \\ \frac{6}{T^2} & \frac{4}{T} \end{pmatrix} \]

\[ u(t) = (0, 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W(0, T)^{-1} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \]

From this it is easy to see that

\[ \eta = \int_0^T u(s)^2 ds \]

\[ = \frac{12 \cos^2(\theta)}{T^3} - \frac{12 \sin(\theta) \cos(\theta)}{T^2} + \frac{4 \sin^2(\theta)}{T} . \]

3(b). Hence

\[ \eta(0) = \eta(\pi) = \frac{12}{T^3}, \quad \text{and} \quad \eta(\pi/2) = \eta(3\pi/2) = \frac{4}{T}. \]

Note that a general form of the optimal cost of steering

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

from \( x_0 \) to \( x_1 \) so as to minimize

\[ \eta = \int_0^T \|u(t)\|^2 dt. \]

is

\[ \eta_0 = [x_0 - e^{-AT}x_1]^T W(0, T)^{-1} [x_0 - e^{-AT}x_1]. \]

The values above can be read off directly from this formula.