

(11/29/11)

**Grading:**

ENG EC/ME/SC 501:

**Exercises (Set 7)** (Due 12/6/11)1. Given that  $x(0) = 1$ , find  $x(\cdot)$  on the interval  $0 \leq t \leq T$  such that**4 pts.**

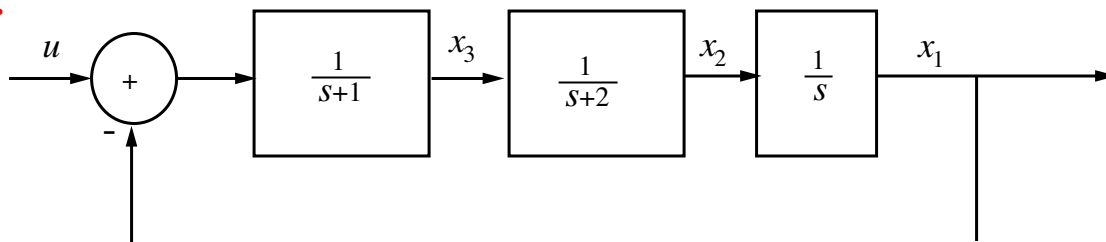
$$J = \int_0^T \dot{x}^2 + x^2 dt$$

is minimized. (*Hint:* Convert this into a control problem by setting  $\dot{x} = u$ .)

2. Suppose that the partitioned system

**3 pts.**

$$\begin{pmatrix} \dot{w}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} w(t) \\ y(t) \end{pmatrix}$$

with output  $y(t)$  is observable. Show that  $\{A_{11}, A_{21}\}$  is an observable pair.3. Design an observer for the system shown in the figure. The observer should be of second order with both eigenvalues equal to  $-3$ .**3 pts.**

## Exercise Set 7 Solutions

1

Allow  $\dot{x} = u$ . Note that this represents a controllable system. Then

$$J = \int_0^T \dot{x}^2 + x^2 dt = \int_0^T u^2 + x^2 dt.$$

This is equivalent to

$$J = \int_0^T \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} dt,$$

where  $\mathbf{Q} = \mathbf{R} = 1$ . Note that both  $\mathbf{Q}$  and  $\mathbf{R}$  are positive definite.

The associated Riccati equation is

$$\dot{m} = (m + 1)(m - 1), \tag{1}$$

which is separable, simplifying to the equation of integrals

$$\int \frac{dm}{(m + 1)(m - 1)} = \int dt. \tag{2}$$

The integral on the right hand side of (2) produces  $t + c$ . The integral on the left hand side of (2), after consulting an integration table, evaluates to

$$\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right).$$

Thus, equation (2) becomes

$$\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right) = t + c,$$

which, after some algebra, produces the solution to (1) as

$$m(t) = \frac{1 + ce^{2t}}{1 - ce^{2t}}. \tag{3}$$

In order for  $m(\cdot)$  to pass through 0 at time  $t = T$ ,  $c$  must be  $-e^{-2T}$ . Accordingly, the solution in (3) becomes

$$m(t) = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T)}} = -\tanh(t - T). \tag{4}$$

Since  $m(t) \geq 0$  for all  $t \in [0, T]$ ,  $m(t)$  is positive semidefinite on the interval  $[0, T]$ . Therefore, the control that minimizes  $J$  is given by

$$u(t) = x(t) \tanh(t - T). \tag{5}$$

Thus  $x(\cdot)$  satisfies

$$\dot{x}(t) = x(t) \tanh(t - T).$$

Solving for  $x(\cdot)$ ,

$$\begin{aligned} \frac{dx}{dt} &= x \tanh(t - T) \\ \int \frac{dx}{x} &= \int \tanh(t - T) dt \\ \ln x &= \ln \cosh(t - T) + c \\ x(t) &= c \cosh(t - T). \end{aligned}$$

Employing the initial condition produces

$$\begin{aligned} x(0) = 1 &= c \cosh(-T) = c \cosh(T) \\ c &= \operatorname{sech}(T). \end{aligned}$$

And thus,

$$x(t) = \operatorname{sech}(T) \cosh(t - T).$$

2

We have

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{A}_{11}\mathbf{w} + \mathbf{A}_{12}\mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{A}_{21}\mathbf{w} + \mathbf{A}_{22}\mathbf{y}. \end{aligned}$$

Allow  $\mathbf{z} = \dot{\mathbf{y}} - \mathbf{A}_{22}\mathbf{y}$ . Since  $\mathbf{y}$  is observable,  $\mathbf{z}$  is a known quantity.

The solution for  $\mathbf{w}$  is obtained by

$$\mathbf{w}(t) = e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f(\mathbf{y})$$

where  $f$  is some function of observable  $\mathbf{y}$  and therefore evaluates to a known quantity. This gives

$$\mathbf{z}(t) = \mathbf{A}_{21}e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f'(\mathbf{y}),$$

where  $f'$  is some other function of observable  $\mathbf{y}$  and therefore evaluates to a known quantity. Differentiating  $\mathbf{z}$  produces

$$\dot{\mathbf{z}}(t) = \mathbf{A}_{21}\mathbf{A}_{11}e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f''(\mathbf{y}),$$

where again  $f''$  is some function that evaluates to a known quantity. Eventually, after repeated differentiation and evaluation, we obtain

$$\begin{bmatrix} \mathbf{z}(0) \\ \dot{\mathbf{z}}(0) \\ \vdots \\ \mathbf{z}^{(n)}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21}\mathbf{A}_{11} \\ \vdots \\ \mathbf{A}_{21}\mathbf{A}_{11}^{n-1} \end{bmatrix} \mathbf{w}_0 + \mathbf{F}(\mathbf{y}),$$

where  $\mathbf{F}$  is some vector that we can assume is known. Since  $\mathbf{F}$ ,  $\mathbf{w}_0$ , and  $[\mathbf{z}(0), \dot{\mathbf{z}}(0), \dots, \mathbf{z}^{(n)}(0)]^T$  are all known quantities, the matrix

$$\begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21}\mathbf{A}_{11} \\ \vdots \\ \mathbf{A}_{21}\mathbf{A}_{11}^{n-1} \end{bmatrix}$$

is invertible, and therefore full rank. Thus,  $(\mathbf{A}_{11}, \mathbf{A}_{21})$  is an observable pair.

3

The system diagram produces the equations

$$\begin{aligned} x_1 &= \frac{1}{s}x_2 \\ x_2 &= \frac{1}{s+2}x_3 \\ x_3 &= \frac{1}{s+1}(-x_1 + u) \\ y &= x_1. \end{aligned}$$

After some rearrangement, this becomes

$$\begin{aligned} sx_1 &= x_2 \\ sx_2 &= -2x_2 + x_3 \\ sx_3 &= -x_1 - x_3 + u \\ y &= x_1, \end{aligned}$$

which produces

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Further rearranging the above produces

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

with partitions

$$\mathbf{A}_{11} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad \mathbf{A}_{12} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \mathbf{A}_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{A}_{22} = \begin{bmatrix} 0 \end{bmatrix}.$$

Hence,

$$\mathbf{A}_{11} - \mathbf{EA}_{21} = \begin{bmatrix} -2 - e_1 & 1 \\ -e_2 & -1 \end{bmatrix},$$

with characteristic polynomial

$$|\mathbf{A}_{11} - \mathbf{EA}_{21} - \mathbf{I}\lambda| = \begin{vmatrix} -2 - e_1 - \lambda & 1 \\ -e_2 & -1 - \lambda \end{vmatrix} = \lambda^2 + (3 + e_1)\lambda + e_1 + e_2 + 2.$$

For both eigenvalues of  $\mathbf{A}_{11} - \mathbf{EA}_{21}$  to be -3 it must be the case that

$$3 + e_1 = 6 \Rightarrow e_1 = 3$$

and

$$e_1 + e_2 + 2 = 9 \Rightarrow e_2 = 4.$$

The observer equation is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -11 \\ -17 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The estimate  $\hat{\mathbf{x}}$  of  $\mathbf{x}$  is then reconstructed as

$$\begin{aligned} \mathbf{P}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ y \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 + e_1 y \\ z_2 + e_2 y \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 + 3y \\ z_2 + 4y \\ y \end{bmatrix} \\ &= \begin{bmatrix} y \\ z_1 + 3y \\ z_2 + 4y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} y. \end{aligned}$$