

(11/01/11)

**Grading plan:**

ENG ME/EC/SE 501:

**Exercises (Set 6)** (Due 11/08/11)

1. The dynamics of a cd-motor are given by  $J\ddot{\theta} + c\dot{\theta} = u(t)$ , where  $J$  is the motor inertia,  $c$  is a coefficient representing forces opposing the motion (back EMF, viscous friction, etc.), and  $u(t)$  is a control input proportional to the applied current. Find the feedback gains in a control law of the form  $u = -k_v\dot{\theta} - k_p\theta$  such that the natural undamped frequency of the system ( $\omega$ ) is 1, and the damping ratio,  $\zeta$  is also 1. ( $k_p$  and  $k_v$  are to be expressed in terms of  $J$  and  $c$ .)

2. Find a control law  $u(\cdot)$  which steers the state of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$$

from  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  in one unit of time.

3. Identify the controllable pairs

$$(i) \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(ii) \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(iii) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

3 pts.

4 pts.

3 pts

## Exercise Set 6

[1]

For the system dynamics  $J\ddot{\theta} + c\dot{\theta} = u(t)$ , where  $u(t) = -k_v\dot{\theta} - k_p\theta$ , the closed loop equation is

$$J\ddot{\theta} + (c + k_v)\dot{\theta} + k_p\theta = 0. \quad (1)$$

From (1), we can obtain the natural undamped frequency,  $\omega$ , and the damping ratio,  $\zeta$ , by

$$\omega^2 = \frac{k_p}{J}$$

and

$$\zeta = \frac{c + k_v}{2\omega J}.$$

For both the natural undamped frequency and the damping ratio to be 1, the equations

$$\frac{k_p}{J} = 1$$

and

$$\frac{c + k_v}{2J} = 1$$

must be satisfied. This occurs when  $k_p = J$  and  $k_v = 2J - c$ .

[2]

There exists a control  $u(\cdot)$  which steers the state  $x(t)$  of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix} \quad (2)$$

from  $\mathbf{x}_0 = [0, 0]^T$  to  $\mathbf{x}_1 = [1, 0]^T$  if and only if

$$\mathbf{x}_0 - \Phi(0, 1)\mathbf{x}_1 \quad (3)$$

belongs to the range space of  $\mathbf{W}(0, 1)$ .

We have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, k = 2, 3, \dots$$

Therefore,

$$\Phi(t, \sigma) = e^{\mathbf{A}(t-\sigma)} = \mathbf{I} + \mathbf{A}(t - \sigma) = \begin{bmatrix} 1 & t - \sigma \\ 0 & 1 \end{bmatrix}.$$

Also, we have

$$\mathbf{W}(0, 1) = \int_0^1 \Phi(0, t)\mathbf{B}\mathbf{B}^T\Phi^T(0, t)dt = \int_0^1 \begin{bmatrix} t^2 & -t \\ -t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

If (3) belongs to the range space of  $\mathbf{W}(0, 1)$ , then there exists  $\eta \in \mathbb{R}^2$  such that

$$\mathbf{x}_0 - \Phi(0, 1)\mathbf{x}_1 = \mathbf{W}(0, 1)\eta. \quad (4)$$

The condition (4) is equivalent to the system of equations

$$\begin{aligned} \frac{1}{3}\eta_1 - \frac{1}{2}\eta_2 &= -1 \\ -\frac{1}{2}\eta_1 + \eta_2 &= 0 \end{aligned}$$

which has the unique solution  $\eta = [-12, -6]^T$ .

Finally, the control law in question is of the form

$$u(t) = -\mathbf{B}^T \Phi^T(0, t)\eta = -12t + 6.$$

3

For the three parts of this question, we will examine the rank of the matrix

$$\mathbf{W} = [ \mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B} ]. \quad (5)$$

(a) For

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

the matrix in (5) becomes

$$\mathbf{W} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which is rank 1. Therefore, the matrices  $\mathbf{A}$  and  $\mathbf{B}$  do not represent a controllable system.

(b) For

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the matrix in (5) becomes

$$\begin{aligned} \mathbf{W} &= \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

which has full row rank. Therefore, the matrices  $\mathbf{A}$  and  $\mathbf{B}$  represent a controllable system.

(c) For

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

the matrix in (5) becomes

$$\mathbf{W} = \left[ \begin{array}{c} \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \\ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \\ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \end{array} \right] = \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{array} \right],$$

which has full row rank. Therefore, the matrices  $\mathbf{A}$  and  $\mathbf{B}$  represent a controllable system.