Grading plan:

ENG EC/ME/SE 501:

Exercises (Set 5)

1. For which of the following systems is the origin asymptotically stable?

4 pts.

(i) $\ddot{x} + ax + bx = 0$, $a > 0, b > 0$,

(ii) $\ddot{x} + ax + bx = 0$, $a < 0, b > 0$,

(iii) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $a < 0$,

(iv) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

You should, of course, state your reasons.

2. For each of the following polynomials determine how many roots are in the right half-plane:

3 pts

(i) $\lambda^2 - 2\lambda + 1$,

(ii) $\lambda^3 + 4\lambda^2 + 5\lambda + 2$,

(iii) $-2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4$.

3. (i) Show that reversing the order of coefficients (replacing $a_i$ by $a_{n-i}$ in the Routh criterion must give the same result.

(ii) Show that this can be helpful for testing $\lambda^6 + \lambda^5 + 3\lambda^4 + 2\lambda^3 + 4\lambda^2 + a\lambda + 8$, where $a$ is a parameter.

2 pts.

(Please turn over.)
4. By examining the Nyquist locus, determine the range of gains $k$ (if any) such that the following closed-loop system is asymptotically stable.

1 pt.
Exercise Set 5
November 1, 2011

(a) For the system
\[
\ddot{x} + ax + bx = 0, \tag{1}
\]
the characteristic polynomial is
\[
s^2 + as + b. \tag{2}
\]
The roots of (2) are easily obtained from the quadratic formula,
\[
s = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{3}
\]
For the case in which \(a > 0\) and \(b > 0\), we have
\[-a < \Re(\sqrt{a^2 - 4b}) < a.\]
Subtracting \(\Re(\sqrt{a^2 - 4b})\) provides
\[-a - \Re(\sqrt{a^2 - 4b}) < 0 < a - \Re(\sqrt{a^2 - 4b}),\]
which provides the inequalities \(-a - \Re(\sqrt{a^2 - 4b}) < 0\) and \(-a + \Re(\sqrt{a^2 - 4b}) < 0\). Thus, the system (1) has all of its roots in the open left half-plane and is therefore asymptotically stable at the origin.

(b) For the case in which \(a < 0\) and \(b > 0\),
\[a < \Re(\sqrt{a^2 - 4b}) < -a.\]
Subtracting \(\Re(\sqrt{a^2 - 4b})\) provides
\[a - \Re(\sqrt{a^2 - 4b}) < 0 < -a - \Re(\sqrt{a^2 - 4b}),\]
which provides the inequalities \(0 < -a - \Re(\sqrt{a^2 - 4b})\) and \(0 < -a + \Re(\sqrt{a^2 - 4b})\). Thus, the system (1) has all of its roots in the right half-plane and is therefore not asymptotically stable at the origin.

(c) The matrix \(A\) is defined as
\[
A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = B + C.
\]
Note,
\[
BC = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ -ab & 0 \end{bmatrix} = CB.
\]
So,
\[e^{At} = e^{Bt} e^{Ct}.\]
The value of the first term is given by
\[ e^{Bt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}. \]

Regarding the second term,
\[ e^{Ct} = \sum_{n=0}^{\infty} C^n \frac{t^n}{n!}. \]

The value of \( C^n \) is given by,
\[ C^{2n} = \begin{bmatrix} (-1)^n b^{2n} & 0 \\ 0 & (-1)^n b^{2n} \end{bmatrix} \]
and
\[ C^{2n+1} = \begin{bmatrix} 0 & (-1)^n b^{2n+1} \\ (-1)^{n+1} b^{2n+1} & 0 \end{bmatrix}, \]
for \( n = 0, 1, \ldots \). Thus,
\[ e^{Ct} = \sum_{n=0}^{\infty} \begin{bmatrix} (-1)^n b^{2n} \frac{t^{2n}}{(2n)!} & (-1)^n b^{2n+1} \frac{t^{2n+1}}{(2n+1)!} \\ (-1)^{n+1} b^{2n+1} \frac{t^{2n}}{(2n)!} & (-1)^n b^{2n+1} \frac{t^{2n+1}}{(2n+1)!} \end{bmatrix}. \]

This gives,
\[ e^{Ct} = \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}. \]

Therefore,
\[ e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix} = \begin{bmatrix} e^{at} \cos bt & e^{at} \sin bt \\ -e^{at} \sin bt & e^{at} \cos bt \end{bmatrix}. \]

If \( a < 0 \), the elements of \( e^{At} \) tend to zero. Therefore, the system
\[ \dot{x} = Ax \]
is asymptotically stable at the origin.

(d) The matrix
\[ A = \begin{bmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{bmatrix} \]
satisfies \( A = B + C \) where
\[ B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]
and
\[ C = \begin{bmatrix} 0 & -a & a \\ a & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}. \]

Note,
\[ BC = -C = CB. \]
Therefore,
\[ e^{At} = e^{Bt} e^{Ct}. \]

Here,
\[ e^{Bt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \]

and
\[ e^{Ct} = \sum_{n=0}^{\infty} C^n t^n/n! \]

Values of \( C^n \) are given by
\[
C^{2n} = \begin{bmatrix} \frac{(-1)^n}{2} (2a)^{2n} \\ 0 \\ 0 \end{bmatrix}, \quad C^{2n+1} = \begin{bmatrix} 0 \\ \frac{(-1)^{n+1}}{4} (2a)^{2n+1} \\ 0 \end{bmatrix},
\]
for \( n = 0, 1, \ldots \). Therefore,
\[
e^{Ct} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{(-1)^n}{2} (2a)^{2n} \\ \frac{(-1)^{n+1}}{4} (2a)^{2n+1} \\ \frac{(-1)^n}{4} (2a)^{2n} \end{bmatrix} \begin{bmatrix} \frac{(-1)^{n+1}}{4} (2a)^{2n+1} \\ \frac{(-1)^n}{4} (2a)^{2n+1} \\ \frac{(-1)^n}{4} (2a)^{2n} \end{bmatrix} = \begin{bmatrix} \frac{\cos(2at)}{2} & -\frac{\sin(2at)}{4} & \frac{\sin(2at)}{4} \\ \frac{-\sin(2at)}{4} & \frac{\cos(2at)}{4} & -\frac{\cos(2at)}{4} \\ \frac{\sin(2at)}{4} & \frac{-\cos(2at)}{4} & \frac{\cos(2at)}{4} \end{bmatrix}.
\]

So,
\[ e^{At} = \begin{bmatrix} e^{-t} \cos(2at) & -\frac{e^{-t} \sin(2at)}{4} & \frac{e^{-t} \sin(2at)}{4} \\ \frac{e^{-t} \sin(2at)}{4} & e^{-t} \cos(2at) & -\frac{e^{-t} \cos(2at)}{4} \\ -\frac{e^{-t} \sin(2at)}{4} & \frac{e^{-t} \cos(2at)}{4} & e^{-t} \cos(2at) \end{bmatrix}. \]

Since all of the elements of \( e^{At} \) tend to zero as \( t \) tends to infinity, the system
\[ \dot{x} = Ax \]

is asymptotically stable at the origin.

2

(a) \( p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \) has a repeated root, \( \lambda = 1 \). Hence, \( p(\lambda) \) has a repeated root in the right half-plane.

(b) \( p(\lambda) = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2 \) has a root \( \lambda = -2 \) and a repeated root \( \lambda = -1 \). Hence, \( p(\lambda) \) has no roots in the right half-plane.
(c) Using the Routh algorithm for \( p(\lambda) = -2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4 \)

\[
\begin{array}{ccc}
4 & 2 & -4 \\
1 & 1 & -2 \\
-2 & 4 \\
3 & -2 \\
\frac{8}{3} & \\
-2 \\
\end{array}
\]

which has 3 sign changes in the first column, implying that \( p(\lambda) \) has 3 roots in the right half-plane.

(a) Consider

\[ p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0. \]  \hspace{1cm} (4)

\( \lambda_0 \) is a right half-plane root of (4) if and only if \( z_0 = \frac{1}{\lambda_0} \) is a right half-plane root of

\[ a_n \left( \frac{1}{z} \right)^n + a_{n-1} \left( \frac{1}{z} \right)^{n-1} + \ldots + a_0 = 0. \]

Equivalently, \( z_0 \) is a right half-plane root of

\[ a_n + a_{n-1}z + \ldots + a_0z^n = 0. \]  \hspace{1cm} (5)

(b) The Routh table

\[
\begin{array}{ccc}
8 & 4 & 3 & 1 \\
\frac{4a-16}{a} & \frac{3a-8}{a} & 1 \\
\vdots & \vdots \\
1 & 3 & 4 & 8 \\
1 & 2 & a \\
\end{array}
\]

gets quite complicated. But, complementing the indices, the Routh table is

\[
\begin{array}{ccc}
1 & 3 & 4 & 8 \\
1 & 2 & a \\
\frac{a^2-5a}{a^2} & a-8 \\
\frac{a^2-80a-32}{a^2-5a} & 8 \\
\end{array}
\]

The complicated algebraic expressions appear later in the Routh table, making it a little easier to handle parametrically.
4. You may use software for this problem. Consider the transfer function

\[ g(s) = \frac{1}{s^3 + s^2 + s + 2}. \]

(i) How many right half-plane poles are there.

*Answer:* 2 \( s = -1.35321, s = 0.176605 \pm 1.20282i \).

(ii) By examining the Nyquist locus, determine the range of gains \( k \) (if any) such that the following closed-loop system is asymptotically stable.

*Answer:* For \( k \) in the interval \(-2 < k < -1\), the root locus encircles \(-1/k\) \(-2\) times in the clockwise sense. Hence in the interval, there are no rhp roots.

(iii) Show that the closed loop poles of this system are the zeros of the polynomial \( p(s) = s^3 + s^2 + s + 2 + k \). Plot the root locus, indicating the parameter ranges (ranges of \( k \)) such that the closed loop system is stable.

*Answer:* The closed-loop poles are the zeros of \( 1 + kg(s) = 1 + k/s^3 + s^2 + s + 2 \). These coincide with the zeros of the polynomial \( p(s) = s^3 + s^2 + s + 2 + k \). See next page.

(iv) Write down the Routh table for \( p(s) \).
The above gives the three branches of the root locus of \( s^3 + s^2 + s + 2 + k = 0 \). All three roots are in the left half plane for \(-2 < k < -1\). For \( k > -1 \), the two complex roots move onto the rhp, while for \( k < -2 \), the real root lies in the rhp.

Problem 4 (iii) solution.