ENG EC/ME/SE 501:

Exercises (Set 5)  (Due 11/01/12)

1. For which of the following systems is the origin asymptotically stable?

\[ (i) \ddot{x} + ax + bx = 0, \quad a > 0, b > 0, \]
\[ (ii) \ddot{x} + ax + bx = 0, \quad a < 0, b > 0, \]
\[ (iii) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a < 0, \]
\[ (iv) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \]

You should, of course, state your reasons.

2. For each of the following polynomials determine how many roots are in the right half-plane:

\[ (i) \lambda^2 - 2\lambda + 1, \]
\[ (ii) \lambda^3 + 4\lambda^2 + 5\lambda + 2, \]
\[ (iii) -2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4. \]

3. (i) Show that reversing the order of coefficients (replacing \( a_i \) by \( a_{n-i} \) in the Routh criterion) must give the same result.

(ii) Show that this can be helpful for testing \( \lambda^6 + \lambda^5 + 3\lambda^4 + 2\lambda^3 + 4\lambda^2 + a\lambda + 8 \), where \( a \) is a parameter.

(Please turn over.)
4. You may use software for this problem. Consider the transfer function

\[ g(s) = \frac{1}{s^3 + s^2 + s + 2}. \]

(i) How many right half-plane poles are there.

(ii) By examining the Nyquist locus, determine the range of gains \( k \) (if any) such that the following closed-loop system is asymptotically stable.

(iii) Show that the closed loop poles of this system are the zeros of the polynomial \( p(s) = s^3 + s^2 + s + 2 + k \). Plot the root locus, indicating the parameter ranges (ranges of \( k \)) such that the closed loop system is stable.

(iv) Write down the Routh table for \( p(s) \).
For the system
\[ \ddot{x} + ax + bx = 0, \tag{1} \]
the characteristic polynomial is
\[ s^2 + as + b. \tag{2} \]
The roots of (2) are easily obtained from the quadratic formula,
\[ s = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{3} \]
For the case in which \( a > 0 \) and \( b > 0 \), we have
\[ -a < \Re(\sqrt{a^2 - 4b}) < a. \]
Subtracting \( \Re(\sqrt{a^2 - 4b}) \) provides
\[ -a - \Re(\sqrt{a^2 - 4b}) < 0 < a - \Re(\sqrt{a^2 - 4b}), \]
which provides the inequalities \(-a - \Re(\sqrt{a^2 - 4b}) < 0 \) and \(-a + \Re(\sqrt{a^2 - 4b}) < 0 \). Thus, the system (1) has all of its roots in the open left half-plane and is therefore asymptotically stable at the origin.

(b) For the case in which \( a < 0 \) and \( b > 0 \),
\[ a < \Re(\sqrt{a^2 - 4b}) < -a. \]
Subtracting \( \Re(\sqrt{a^2 - 4b}) \) provides
\[ a - \Re(\sqrt{a^2 - 4b}) < 0 < -a - \Re(\sqrt{a^2 - 4b}), \]
which provides the inequalities \( 0 < -a - \Re(\sqrt{a^2 - 4b}) \) and \( 0 < -a + \Re(\sqrt{a^2 - 4b}) \). Thus, the system (1) has all of its roots in the right half-plane and is therefore not asymptotically stable at the origin.

(c) The matrix \( A \) is defined as
\[ A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = B + C. \]
Note,
\[ BC = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ -ab & 0 \end{bmatrix} = CB. \]
So,
\[ e^{At} = e^{Bt} e^{Ct}. \]
The value of the first term is given by
\[ e^{Bt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}. \]

Regarding the second term,
\[ e^{Ct} = \sum_{n=0}^{\infty} C^n t^n / n!. \]

The value of \( C^n \) is given by,
\[ C^{2n} = \begin{bmatrix} (-1)^n b^{2n} & 0 \\ 0 & (-1)^n b^{2n} \end{bmatrix} \]
and
\[ C^{2n+1} = \begin{bmatrix} 0 & (-1)^n b^{2n+1} \\ (-1)^n b^{2n+1} & 0 \end{bmatrix}, \]
for \( n = 0, 1, \ldots \) Thus,
\[ e^{Ct} = \sum_{n=0}^{\infty} \begin{bmatrix} (-1)^n b^{2n} \frac{t^{2n}}{(2n)!} & (-1)^n b^{2n+1} \frac{t^{2n+1}}{(2n+1)!} \\ (-1)^n b^{2n+1} \frac{t^{2n}}{(2n)!} & (-1)^n b^{2n} \frac{t^{2n+1}}{(2n+1)!} \end{bmatrix}. \]
This gives,
\[ e^{Ct} = \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}. \]

Therefore,
\[ e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix} = \begin{bmatrix} e^{at} \cos bt & e^{at} \sin bt \\ -e^{at} \sin bt & e^{at} \cos bt \end{bmatrix}. \]

If \( a < 0 \), the elements of \( e^{At} \) tend to zero. Therefore, the system
\[ \dot{x} = Ax \]
is asymptotically stable at the origin.

(d) The matrix
\[ A = \begin{bmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{bmatrix} \]
satisfies \( A = B + C \) where
\[ B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]
and
\[ C = \begin{bmatrix} 0 & -a & a \\ a & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}. \]

Note,
\[ BC = -C = CB. \]
Therefore, 
\[ e^{At} = e^{Bt} e^{Ct}. \]

Here, 
\[ e^{Bt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \]
and
\[ e^{Ct} = \sum_{n=0}^{\infty} C^n t^n/n! . \]
Values of \( C^n \) are given by
\[ C^{2n} = \begin{bmatrix} (-1)^n (2a)^{2n} \frac{1}{2} & 0 & (-1)^n (2a)^{2n} \frac{1}{4} \\ 0 & (-1)^{n+1} (2a)^{2n} \frac{1}{4} & (-1)^n (2a)^{2n} \frac{1}{4} \\ 0 & 0 & (-1)^n (2a)^{2n} \frac{1}{4} \end{bmatrix} \]
and
\[ C^{2n+1} = \begin{bmatrix} (-1)^n (2a)^{2n+1} \frac{1}{4} & 0 & (-1)^n (2a)^{2n+1} \frac{1}{4} \\ 0 & (-1)^{n+1} (2a)^{2n+1} \frac{1}{4} & (-1)^n (2a)^{2n+1} \frac{1}{4} \\ (-1)^n (2a)^{2n+1} \frac{1}{4} & 0 & (-1)^n (2a)^{2n+1} \frac{1}{4} \end{bmatrix} , \]
for \( n = 0, 1, \ldots \). Therefore,
\[ e^{Ct} = \sum_{n=0}^{\infty} \begin{bmatrix} (-1)^n (2a)^{2n} \frac{1}{2} & 0 & (-1)^n (2a)^{2n} \frac{1}{4} \\ 0 & (-1)^{n+1} (2a)^{2n+1} \frac{1}{4} & (-1)^n (2a)^{2n+1} \frac{1}{4} \\ 0 & 0 & (-1)^n (2a)^{2n+1} \frac{1}{4} \end{bmatrix} \begin{bmatrix} \cos(2at) \\ \frac{2}{\sin(2at)} \cos(2at) \\ \frac{1}{\sin(2at)} \cos(2at) \end{bmatrix} = \begin{bmatrix} \cos(2at) \\ \frac{2}{\sin(2at)} \cos(2at) \\ \frac{1}{\sin(2at)} \cos(2at) \end{bmatrix} . \]

So,
\[ e^{At} = \begin{bmatrix} e^{-t} \cos(2at) & \frac{e^{-t} \sin(2at)}{4} & \frac{-e^{-t} \sin(2at)}{4} \\ \frac{-e^{-t} \sin(2at)}{4} & e^{-t} \cos(2at) & \frac{-e^{-t} \cos(2at)}{4} \\ \frac{-e^{-t} \sin(2at)}{4} & \frac{-e^{-t} \cos(2at)}{4} & e^{-t} \cos(2at) \end{bmatrix} . \]
Since all of the elements of \( e^{At} \) tend to zero as \( t \) tends to infinity, the system
\[ \dot{x} = Ax \]
is asymptotically stable at the origin.

[2]

(a) \( p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \) has a repeated root, \( \lambda = 1 \). Hence, \( p(\lambda) \) has a repeated root in the right half-plane.

(b) \( p(\lambda) = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2 \) has a root \( \lambda = -2 \) and a repeated root \( \lambda = -1 \). Hence, \( p(\lambda) \) has no roots in the right half-plane.
(c) Using the Routh algorithm for 
\[ p(\lambda) = -2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4 \]

\[
\begin{array}{ccc}
4 & 2 & -4 \\
1 & 1 & -2 \\
-2 & 4 \\
3 & -2 \\
\frac{8}{3} & \\
-2 \\
\end{array}
\]

which has 3 sign changes in the first column, implying that \( p(\lambda) \) has 3 roots in the right half-plane.

(a) Consider
\[ p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0. \]  
\( \lambda_0 \) is a right half-plane root of (4) if and only if \( z_0 = \frac{1}{\lambda_0} \) is a right half-plane root of
\[ a_n \left( \frac{1}{z} \right)^n + a_{n-1} \left( \frac{1}{z} \right)^{n-1} + \ldots + a_0 = 0. \]

Equivalently, \( z_0 \) is a right half-plane root of
\[ a_n + a_{n-1}z + \ldots + a_0z^n = 0. \]

(b) The Routh table

\[
\begin{array}{cccc}
8 & 4 & 3 & 1 \\
\frac{4a-16}{a} & \frac{3a-8}{a} & 1 \\
\end{array}
\]

gets quite complicated. But, complementing the indices, the Routh table is

\[
\begin{array}{cccc}
1 & 3 & 4 & 8 \\
1 & 2 & a \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 4 - a & 8 \\
ap - 2 & a - 8 \\
a^2 - 5a & 8 \\
a^2 - 21a^2 + 80a - 32 & \frac{8}{a^2 - 5a} \\
\end{array}
\]

The complicated algebraic expressions appear later in the Routh table, making it a little easier to handle parametrically.
4. You may use software for this problem. Consider the transfer function

\[ g(s) = \frac{1}{s^3 + s^2 + s + 2}. \]

(i) How many right half-plane poles are there.

\[ s = -1.35321, s = 0.176605 \pm 1.20282i. \]

(ii) By examining the Nyquist locus, determine the range of gains \( k \) (if any) such that the following closed-loop system is asymptotically stable.

\[ \text{Answer:} \]

\[ \begin{array}{c}
\text{Answer:} \\
\end{array} \]

(iii) Show that the closed loop poles of this system are the zeros of the polynomial \( p(s) = s^3 + s^2 + s + 2 + k \). Plot the root locus, indicating the parameter ranges (ranges of \( k \)) such that the closed loop system is stable.

\[ \text{Answer:} \] The closed-loop poles are the zeros of \( 1 + kg(s) = 1 + k/s^3 + s^2 + s + 2 \). These coincide with the zeros of the polynomial \( p(s) = s^3 + s^2 + s + 2 + k \).

(iv) Write down the Routh table for \( p(s) \).
$\phi(s) = s^3 + s^2 + s + z + k$

\[
\begin{array}{c|c|c|c}
-z - k & 1 & 1 \\
1 & 1 & 1 \\
1 - (2 + k) & 0 & 0 \\
-k - 1 & -k - 1 & -k - 1 \\
\end{array}
\]

The left hand column is where we look for sign change.

$2+k$

$1$

$-1-k$

$1$

The first entry $w > 0 \iff k > -2$. The third is $> 0 \iff k < -1$. Hence, all entries are positive $\iff -2 < k < -1$. If $k < -2$, there is one sign change and hence one r.h.p. zero. If $k > -1$, there are 2 sign changes, and hence, 2 r.h.p. zeros.