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(10/20/11)

`Grading plan:`

ENG EC/ME/SE 501:

Exercises (Set 5) (Due 10/27/11)

`4 pts.`

1. For which of the following systems is the origin asymptotically stable?

(i) $\ddot{x} + a\dot{x} + bx = 0, \quad a > 0, b > 0,$

(ii) $\ddot{x} + a\dot{x} + bx = 0, \quad a < 0, b > 0,$

$$(iii) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad a < 0,$$

$$(iv) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

You should, of course, state your reasons.

2. For each of the following polynomials determine how many roots are in the right half-plane:

`3 pts`

(i) $\lambda^2 - 2\lambda + 1,$

(ii) $\lambda^3 + 4\lambda^2 + 5\lambda + 2,$

(iii) $-2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4.$

3. (i) Show that reversing the order of coefficients (replacing a_i by a_{n-i} in the Routh criterion must give the same result.

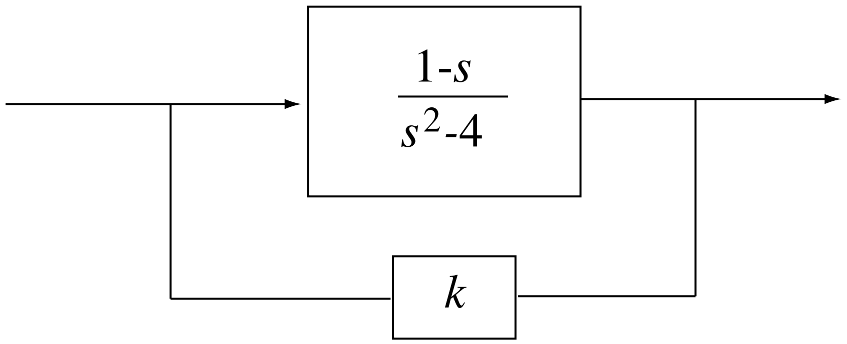
(ii) Show that this can be helpful for testing $\lambda^6 + \lambda^5 + 3\lambda^4 + 2\lambda^3 + 4\lambda^2 + a\lambda + 8$, where a is a parameter.

`2 pts.`

`(Please turn over.)`

4. By examining the Nyquist locus, determine the range of gains k (if any) such that the following closed-loop system is asymptotically stable.

1 pt.



Exercise Set 5

November 1, 2011

$\boxed{1}$

(a) For the system

$$\ddot{x} + a\dot{x} + bx = 0, \tag{1}$$

the characteristic polynomial is

$$s^2 + as + b. \tag{2}$$

The roots of (2) are easily obtained from the quadratic formula,

$$s = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{3}$$

For the case in which $a > 0$ and $b > 0$, we have

$$-a < \Re(\sqrt{a^2 - 4b}) < a.$$

Subtracting $\Re(\sqrt{a^2 - 4b})$ provides

$$-a - \Re(\sqrt{a^2 - 4b}) < 0 < a - \Re(\sqrt{a^2 - 4b}),$$

which provides the inequalities $-a - \Re(\sqrt{a^2 - 4b}) < 0$ and $-a + \Re(\sqrt{a^2 - 4b}) < 0$. Thus, the system (1) has all of its roots in the open left half-plane and is therefore asymptotically stable at the origin.

(b) For the case in which $a < 0$ and $b > 0$,

$$a < \Re(\sqrt{a^2 - 4b}) < -a.$$

Subtracting $\Re(\sqrt{a^2 - 4b})$ provides

$$a - \Re(\sqrt{a^2 - 4b}) < 0 < -a - \Re(\sqrt{a^2 - 4b}),$$

which provides the inequalities $0 < -a - \Re(\sqrt{a^2 - 4b})$ and $0 < -a + \Re(\sqrt{a^2 - 4b})$. Thus, the system (1) has all of its roots in the right half-plane and is therefore not asymptotically stable at the origin.

(c) The matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \mathbf{B} + \mathbf{C}.$$

Note,

$$\mathbf{BC} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ -ab & 0 \end{bmatrix} = \mathbf{CB}.$$

So,

$$e^{\mathbf{A}t} = e^{\mathbf{B}t} e^{\mathbf{C}t}.$$

The value of the first term is given by

$$e^{\mathbf{B}t} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}.$$

Regarding the second term,

$$e^{\mathbf{C}t} = \sum_{n=0}^{\infty} \mathbf{C}^n \frac{t^n}{n!}.$$

The value of $\mathbf{C}^n$ is given by,

$$\mathbf{C}^{2n} = \begin{bmatrix} (-1)^n b^{2n} & 0 \\ 0 & (-1)^n b^{2n} \end{bmatrix}$$

and

$$\mathbf{C}^{2n+1} = \begin{bmatrix} 0 & (-1)^n b^{2n+1} \\ (-1)^{n+1} b^{2n+1} & 0 \end{bmatrix},$$

for $n = 0, 1, \ldots$. Thus,

$$e^{\mathbf{C}t} = \sum_{n=0}^{\infty} \begin{bmatrix} (-1)^n b^{2n} \frac{t^{2n}}{(2n)!} & (-1)^n b^{2n+1} \frac{t^{2n+1}}{(2n+1)!} \\ (-1)^{n+1} b^{2n+1} \frac{t^{2n+1}}{(2n+1)!} & (-1)^n b^{2n} \frac{t^{2n}}{(2n)!} \end{bmatrix}.$$

This gives,

$$e^{\mathbf{C}t} = \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$

Therefore,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix} = \begin{bmatrix} e^{at} \cos bt & e^{at} \sin bt \\ -e^{at} \sin bt & e^{at} \cos bt \end{bmatrix}.$$

If $a < 0$, the elements of $e^{\mathbf{A}t}$ tend to zero. Therefore, the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is asymptotically stable at the origin.

(d) The matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{bmatrix}$$

satisfies $\mathbf{A} = \mathbf{B} + \mathbf{C}$ where

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} 0 & -a & a \\ a & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}.$$

Note,

$$\mathbf{BC} = -\mathbf{C} = \mathbf{CB}.$$

Therefore,

$$e^{\mathbf{A}t} = e^{\mathbf{B}t} e^{\mathbf{C}t}.$$

Here,

$$e^{\mathbf{B}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

and

$$e^{\mathbf{C}t} = \sum_{n=0}^{\infty} \mathbf{C}^n \frac{t^n}{n!}.$$

Values of $\mathbf{C}^n$ are given by

$$\mathbf{C}^{2n} = \begin{bmatrix} \frac{(-1)^n}{2}(2a)^{2n} & 0 & 0 \\ 0 & \frac{(-1)^n}{4}(2a)^{2n} & \frac{(-1)^{n+1}}{4}(2a)^{2n} \\ 0 & \frac{(-1)^{n+1}}{4}(2a)^{2n} & \frac{(-1)^n}{4}(2a)^{2n} \end{bmatrix}$$

and

$$\mathbf{C}^{2n+1} = \begin{bmatrix} 0 & \frac{(-1)^{n+1}}{4}(2a)^{2n+1} & \frac{(-1)^n}{4}(2a)^{2n+1} \\ \frac{(-1)^n}{4}(2a)^{2n+1} & 0 & 0 \\ \frac{(-1)^{n+1}}{4}(2a)^{2n+1} & 0 & 0 \end{bmatrix},$$

for $n = 0, 1, \ldots$. Therefore,

$$e^{\mathbf{C}t} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{(-1)^n}{2}\frac{(2at)^{2n}}{(2n)!} & \frac{(-1)^{n+1}}{4}\frac{(2at)^{2n+1}}{(2n+1)!} & \frac{(-1)^n}{4}\frac{(2at)^{2n+1}}{(2n+1)!} \\ \frac{(-1)^n}{4}\frac{(2at)^{2n+1}}{(2n+1)!} & \frac{(-1)^n}{4}\frac{(2at)^{2n}}{(2n)!} & \frac{(-1)^{n+1}}{4}\frac{(2at)^{2n}}{(2n)!} \\ \frac{(-1)^{n+1}}{4}\frac{(2at)^{2n+1}}{(2n+1)!} & \frac{(-1)^{n+1}}{4}\frac{(2at)^{2n}}{(2n)!} & \frac{(-1)^n}{4}(\frac{(2at)^{2n}}{(2n)!} \end{bmatrix} = \begin{bmatrix} \frac{\cos(2at)}{2} & \frac{-\sin(2at)}{4} & \frac{\sin(2at)}{4} \\ \frac{\sin(2at)}{4} & \frac{\cos(2at)}{4} & \frac{-\cos(2at)}{4} \\ \frac{-\sin(2at)}{4} & \frac{-\cos(2at)}{4} & \frac{\cos(2at)}{4} \end{bmatrix}.$$

So,

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{e^{-t}\cos(2at)}{2} & \frac{-e^{-t}\sin(2at)}{4} & \frac{e^{-t}\sin(2at)}{4} \\ \frac{e^{-t}\sin(2at)}{4} & \frac{e^{-t}\cos(2at)}{4} & \frac{-e^{-t}\cos(2at)}{4} \\ \frac{-e^{-t}\sin(2at)}{4} & \frac{-e^{-t}\cos(2at)}{4} & \frac{e^{-t}\cos(2at)}{4} \end{bmatrix}.$$

Since all of the elements of $e^{\mathbf{A}t}$ tend to zero as t tends to infinity, the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is asymptotically stable at the origin.

$\boxed{2}$

(a) $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ has a repeated root, $\lambda = 1$. Hence, $p(\lambda)$ has a repeated root in the right half-plane.

(b) $p(\lambda) = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2$ has a root $\lambda = -2$ and a repeated root $\lambda = -1$. Hence, $p(\lambda)$ has no roots in the right half-plane.

(c) Using the Routh algorithm for $p(\lambda) = -2\lambda^5 - 4\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 4$

$$\begin{array}{ccc}
4 & 2 & -4 \\
1 & 1 & -2 \\
\hline
 & & \\
-2 & 4 & \\
3 & -2 & \\
\frac{8}{3} & & \\
-2 & &
\end{array}$$

which has 3 sign changes in the first column, implying that $p(\lambda)$ has 3 roots in the right half-plane.

$\boxed{3}$

(a) Consider

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0. \tag{4}$$

λ_0 is a right half-plane root of (4) if and only if $z_0 = \frac{1}{\lambda_0}$ is a right half-plane root of

$$a_n\left(\frac{1}{z}\right)^n + a_{n-1}\left(\frac{1}{z}\right)^{n-1} + \ldots + a_0 = 0.$$

Equivalently, z_0 is a right half-plane root of

$$a_n + a_{n-1}z + \ldots + a_0z^n = 0. \tag{5}$$

(b) The Routh table

$$\begin{array}{cccc}
8 & 4 & 3 & 1 \\
a & 2 & 1 & \\
\hline
\frac{4a-16}{a} & \frac{3a-8}{a} & 1 & \\
\vdots & \vdots & &
\end{array}$$

gets quite complicated. But, complementing the indices, the Routh table is

$$\begin{array}{cccc}
1 & 3 & 4 & 8 \\
1 & 2 & a & \\
\hline
 & & & \\
1 & 4-a & 8 & \\
a-2 & a-8 & & \\
\frac{a^2-5a}{a-2} & 8 & & \\
\frac{a^3-21a^2+80a-32}{a^2-5a} & & & \\
8 & & &
\end{array}$$

The complicated algebraic expressions appear later in the Routh table, making it a little easier to handle parametrically.

$\boxed{4}$

The function

$$g(s) = \frac{1-s}{s^2-4}$$

has one pole in the right half-plane. Also, $g(\cdot)$ has the corresponding Nyquist locus

$$\Gamma(g) = \left\{ \frac{-1}{\omega^2+4} + i\frac{\omega}{\omega^2+4} \right\},$$

which encircles $\frac{-1}{k}$ once in the clockwise direction provided

$$\frac{-1}{4} < \frac{-1}{k} < 0,$$

or

$$k > 4.$$

In which case, there are two right half-plane poles. Further, there is a single right half-plane pole for values of k which satisfy

$$k < 4.$$

Therefore, the system is not stable for any values of k.