INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control (in press)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1142

Robust and efficient quantization and coding for control of multidimensional linear systems under data rate constraints

K. Li[1] and J. Baillieul[2,*,†]

[1] *Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY 14850, U.S.A.*
[2] *Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215, U.S.A.*

SUMMARY

Recently, we reported results on coding strategies for scalar feedback systems with data-rate-limited feedback channels in which the data-rate constraints are time varying. Such rate-varying channels are typically encountered in communication networks in which links between nodes are subject to noise, congestion, and intermittent disruption.

The present paper describes results of extending this research into the multidimensional domain. An important consideration is that for systems of dimension greater than one, many classical feedback designs cannot be realized for operation near the theoretical minimum possible data rate. A novel control coding scheme will be presented, and in terms of this, it will be shown that the advantages of coarse signal quantization that had been reported earlier for scalar systems remain in the multidimensional case. The key is to allocate the communication bandwidth efficiently among faster and slower modes. We discuss various strategies that allocate bandwidth by scheduling the time slots assigned to each mode. In particular, we propose a 'robust attention varying' technique, whose merit will be discussed in terms of its robustness with respect to time-varying communication channel capacity and also in terms of how well it operates when the feedback channel capacity is near the theoretical minimum data rate. Copyright © 2006 John Wiley & Sons, Ltd.

Received 3 June 2005; Accepted 17 October 2005

KEY WORDS: data-rate inequality; regular quantization; attention-varying feedback control

*Correspondence to: J. Baillieul, Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215, U.S.A.
† E-mail: johnb@bu.edu

Contract/grant sponsor: Boston University and the Center for Networked Communicating Control Systems; contract/grant number: DAAD19-01-1-0465
Contract/grant sponsor: The National Science Foundation; contract/grant number: DMI-0330171

1. INTRODUCTION

Consider the scalar plant $G(s) = 1/(s - a)$ where a is a positive real number. In [1] and independently in other settings in [2, 3], it was shown that if a feedback control law was implemented using digital communication and processing components, a closed-loop control could be designed to execute bounded motions if and only if the data rate around the closed loop exceeded $a \cdot \log_2 e$. It is not difficult to see that if a number of such open-loop unstable scalar plants $\{1/(s - a_i) : a_i > 0, 1 \leqslant i \leqslant m\}$ are controlled in parallel, digital control laws can be designed to execute bounded motions precisely when the total channel capacity exceeds $\log_2 e \cdot \sum a_i$ and can be allocated as needed to the different plants in the system. It is a somewhat remarkable result, proved independently by a number of researchers, that for any linear plant having open-loop poles $a_1, \ldots, a_k$ in the right half-plane, stabilizing digital feedback laws can be designed if and only if the feedback loop data rate R satisfies

$$R > \log_2 e \cdot \sum \Re(a_i) \tag{1}$$

(see e.g. [1–8].) The control designs that achieve this bound are different from classical ones in that they consider feedback coding explicitly. The proofs of this 'minimum data-rate' theorem (commonly known as the Data-Rate Theorem) have suggested that under some ideal assumptions, it is immaterial whether one uses control designs with fast sampling rates but a small number of possible control values or designs which involve a large number of (finely quantized) control values but with sampling and data transmission done at a slower rate.

Recently, results have appeared showing that in the case where the feedback channel capacity is time-varying (as it would be in, say, a wireless communication network), we do not have an exact tradeoff between the fineness of temporal and spatial quantization. Indeed, we have shown that in the case of time-varying channel capacity, control designs which operate robustly and satisfactorily over a range of sampling rates must involve short word-length encoding of control inputs (see [9]). More specifically, for scalar systems, one-bit control (two control levels) was shown to provide the best performance for systems in which the feedback channels have time-varying channel capacity. One-bit control coding can be used to provide bounded system response when the data rate through the feedback channel is near the theoretical minimum limit described above while at the same time producing good performance when the channel capacity was high. All finer quantizations of control were shown to be less robust with respect to such variability in channel capacity, and it was further shown that for implementations of quantized control satisfying some mild performance regularity conditions in the face of asynchronism between sampling and actuation, the required data rate for control coding involving more than a single bit was significantly higher. A hierarchical control synthesis was also suggested by Li and Baillieul [9] that supplements the one-bit control with a side channel that is used to adjust the magnitude of the one-bit control. It was shown that asymptotic stability can be achieved as long as the feedback data rate for the one-bit control is strictly higher than the theoretical limit and the side channel data rate is non-zero. Similar ideas have been suggested by Brockett and Liberzon [6] and Fagnani and Zampieri [10], with somewhat different emphasis.

This paper synthesizes and refines earlier preliminary results published in [11, 12]. For completeness, some of the basic constructions used in the analysis (e.g. the *virtual systems*) are repeated here for the sake of clarity. The primary contribution of the present paper is to suggest a feedback control quantization and coding strategy which exhibits and extends the robustness

DOI: 10.1002/rnc

and efficiency of the form found in the previously studied one-bit control of scalar systems. In particular, we shall think of a control strategy as being robust and efficient if

(1) it does not require extremely high data rates in either instantaneous or time-averaged sense,
(2) it allows the system to operate near the theoretical data-rate limit (1) in either the instantaneous or time-averaged sense,
(3) it does not produce a degraded performance when switching from a lower data rate to a higher one, and
(4) it satisfies the above when there is mild delay between sampling and control actuation.

The notions of instantaneous and average data rate will become precise later. The third point was referred to as the *regularity* condition in [9] and will not be the main topic here. But it is worthwhile to note that the one-bit control was shown to be preferable in the light of this condition. The present paper discusses encodings which extend the idea of one-bit control to the multivariable case. Such encodings enjoy the kinds of advantages we saw for single-bit control in the case of scalar systems.

In the present paper we shall revisit the control coding approach we proposed in [12], with the aim of providing a better understanding of how to allocate the communication channel to different modes. The approach is to make use of a novel decoupling scheme combined with block coding of the control signals. Part of the novelty of the proposed control technique is that it assumes a mild form of distributed intelligence in which the controller communicates over a data channel what action the plant should take without any specification of how the action should be taken. As a result of decoupling, the encoding of control signals may be carried out in terms of a *virtual system*, which is a collection of scalar systems being run in parallel. Applying the *virtual system* approach to multidimensional systems, it was seen that a proposed maximally coarse control coding could operate near the theoretical minimum (1) only when all eigenvalues of the open-loop system were of similar magnitude. The problem seemed to be that the proposed coding devoted too much bandwidth to slower modes. We will suggest approaches which schedule the time slots assigned to each mode and reduce the system's overall data-rate requirement in the average (over time) sense. We call these approaches collectively '*attention varying*' because the bandwidth assigned to each mode will be time varying. (The fastest modes, however, will receive nearly constant attention.) In particular, we will discuss a robust attention varying technique, whose merits will be discussed in terms of its robustness with respect to varying feedback channel capacity, and also in terms of how well it operates when the feedback channel capacity is near the theoretical minimum requirement.

Section 2 briefly describes the *virtual system* approach [12] for coding control actions for multidimensional linear systems. Section 3 examines the communication constraints for multidimensional systems based on the *virtual system*. In particular, two data-rate concepts are considered: (1) *base data-rate*, R_b, corresponding to the peak instantaneous data-rate requirement that must be satisfied when all decoupled subsystems require attention; and (2) *average data-rate*, R_a, corresponding to the time-averaged data flow the control system needs. Following the previous results, we will show that for the multidimensional system decoupled using the *virtual system* approach, the control strategy adapted directly from the scalar case (call it the *simple strategy*!) does not generally operate with either R_b or R_a close to the theoretical minimum data rate. In terms of robustness to sampling-actuation asynchronism and disturbances, Section 4 shows (extending the somewhat qualitative evidence in [9]) that using

DOI: 10.1002/rnc

feedback control based on time-slot-based communication bandwidth allocation together with one-bit coding for each mode is preferable to allocate the band-width by allocating quantization levels. In Section 5, we discuss time-slot-based feedback channel allocation techniques, which utilize novel control magnitude protocols. Both an aggressive approach and a robust approach are presented. The aggressive approach can achieve an average data-rate requirement that is arbitrarily close to the theoretical limit. However, it requires good knowledge of the communication data rate. The robust approach is able to operate without specific knowledge of the communication channel, and it allows the multidimensional system to operate with R_a very close to the theoretical minimum data rate. The required R_b under this technique is the same as that of the system with the *simple strategy*. Reducing R_b for multidimensional systems in general seems hard. But we shall indicate that it can be done when the eigenvalues of the system satisfy certain special conditions.

2. A VIRTUAL CONTROL APPROACH FOR MULTIDIMENSIONAL SYSTEMS

In the present paper, we consider the *digital finite communication bandwidth* (DFCB) control ([9]) of the n-dimensional open-loop unstable system

$$\dot{x} = Ax + bu \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is the scalar input.

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

where a_i, $i = 1, \ldots, n$ are the distinct positive eigenvalues. Cases with Jordan blocks of sizes larger than one have been discussed in [13]. It was shown that the tradeoff between spatial and temporal quantizations of control coding becomes more complicated in such cases.‡ As a result, the problem of designing robust and efficient control codings for plants with non-trivial Jordan blocks is better examined in a case-by-case fashion. For this reason, we will only consider the scalar Jordan blocks in what follows.

If (2) is sampled uniformly in time, with sampling interval h, and control actions are applied without delay, then it is equivalent to the discrete-time system

$$x(j+1) = Fx(j) + \Gamma u(j) \tag{3}$$

where

$$F = e^{Ah} \quad \text{and} \quad \Gamma = A^{-1}(e^{Ah} - I)b$$

‡Assume that the control objective is producing bounded response and memoryless control laws are used. For a scalar plant, if no sampling-actuation asynchronism is present, then the data-rate bound (1) can be achieved regardless of the control alphabet size. But for a plant with one non-trivial Jordan block, (1) can only be achieved when the control alphabet size approaches infinity. On the other hand, for both cases, the required data rate increases more significantly for control laws with larger alphabets in the presence of asynchronism (See [13]).

DOI: 10.1002/rnc

In [9, 14], we introduced an asynchronous action model. For the sake of completeness, the essential features of this model will be described in the next section. Both the discrete-time model and the asynchronous action model will describe different parts of a single DFCB control system. The communication between the controller and the plant will be assumed to follow a slightly simplified asynchronous action model. A discrete-time-equivalent open-loop control is used by the plant side to decouple the dynamics of the plant. Details will be discussed as we proceed.

In [12], we introduced a *virtual systems* approach for control coding, which proved to be useful in designing and realizing quantized control of system (2). Here, we restate the approach briefly since later sections will depend on it.

Consider the problem of selecting a control law for (2) such that the state evolution approximates the state evolution of

$$\dot{x} = A(x + v) \tag{4}$$

where $v = (v_1, v_2, \ldots, v_n)$ and $v_i = f_i(x_i),\ \ i = 1, \ldots, n$. Since there is only one scalar control input u in (2), (4) cannot be simulated exactly. However, a sampled version of (4) can be realized through (3).

To each possible value of the vector v, we associate a control input sequence $u_0(v)$, $u_1(v), \ldots, u_{n-1}(v)$, obtained by solving the system of linear equations

$$(e^{Anh} - I)v = F^{n-1}\Gamma u_0 + F^{n-2}\Gamma u_1 + \cdots + \Gamma u_{n-1} \tag{5}$$

For models of the form we are considering, the controllability of (A, b) implies the controllability of the pair (F, Γ), and this in turn implies that the set of vectors

$$\{F^{n-1}\Gamma, \ldots, F\Gamma, \Gamma\}$$

is linearly independent. The control sequence calculated from (5) can be applied to (3) for n steps to yield the state transition

$$x(j + n) = F^n x(j) + F^{n-1}\Gamma u_0 + F^{n-2}\Gamma u_1 + \cdots + \Gamma u_{n-1} \tag{6}$$

which can be equivalently written as

$$x(t_j + nh) = e^{Anh}x(t_j) + (e^{Anh} - I)v \tag{7}$$

It is as if the system has undergone the dynamics of (4) (with a constant v) for nh units of time. When h is small enough, (7) closely approximates (4). Then, the system can be viewed as a parallel interconnection of scalar virtual subsystems

$$\dot{x}_i = a_i(x_i + v_i), \quad i = 1, \ldots, n \tag{8}$$

The quantity $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ is the virtual control vector.

The above approach can be implemented in the DFCB framework as follows: let the admissible control set for each virtual subsystem be

$$\mathscr{V}_i = \{v_1, v_2, \ldots, v_{N_i}\}$$

Then, $v \in \mathscr{V}$, where $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 \times \cdots \times \mathscr{V}_n$ has $\prod_{i=1}^{n} N_i$ elements, each associated with a real (as opposed to virtual) control input sequence calculated from (5). Let these sequences be stored at the plant side and invoked by the corresponding virtual control vector v. The controller and the plant only communicate the quantized state feedback and the coded v. More specifically, when the plant side receives a virtual control vector, it applies the associated control input

DOI: 10.1002/rnc

sequence, with each value in the sequence held for h units of time. When the sequence is finished, if no new virtual control vector arrives, the current sequence is repeated. Otherwise, apply the associated new sequence. Here, we assume the control actuation happens upon the completion of one control sequence (of duration nh). This limitation is mild when h is small. The assumption that h is small is also reasonable since the control sequences are prestored at the plant side. Although control values may be streaming from the controller to the plant as the one sequence is being executed, the model is an idealization in which it is assumed that there is no controller–plant communication during the execution of each sequence.

3. CONTROL CODING AND COMMUNICATION BASED ON THE VIRTUAL SYSTEMS

The structure of a DFCB control system based on the virtual systems approach is illustrated in Figure 1. From this figure, it is clear that although the dynamics of the system is decoupled, the communication constraints on the virtual subsystems are still tied together. Let the communication scheme follow a simplified version of the *asynchronous action* model studied in [9]. Each virtual subsystem can be sampled and actuated only at the time instants specified by the common communication scheme—the *common sampling instants* t_j and the *common* (*virtual*) *control updating instants* $p_j \geqslant t_j$, $j = 0, 1, \ldots$ (*asynchronous* when $p_j > t_j$). In [9], t_j's and p_j's are not necessarily evenly spaced, respectively. The quantity $\Theta = \sup_{j \geqslant 1}\{p_j - t_{j-1}\}$ (called the *control interval*) turns out to be important. It is a measure of how 'stale' the control value can be in the case where sampling and actuation are not performed simultaneously. In [12], the system behaviour for different degrees of asynchronism (measured by $\Omega = \sup_{j \geqslant 1}\{p_j - t_j\}/\sup_{j \geqslant 1}\{p_j - t_{j-1}\}$) was also examined. The ideas of

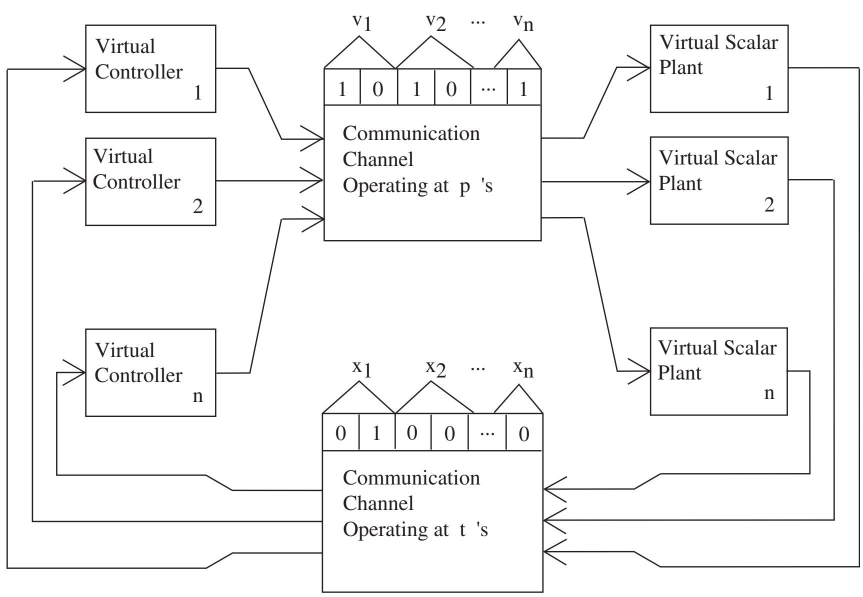

Figure 1. Structure of the DFCB control system based on the virtual systems approach.

DOI: 10.1002/rnc

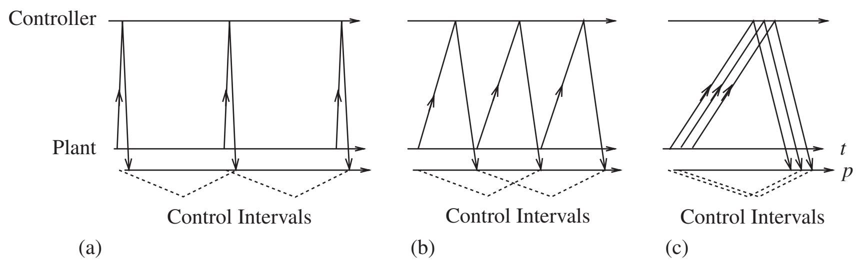

Figure 2. Scenarios of same control interval (Θ) but different degrees of asynchronism (Ω): (a) $\Omega \to 0$; (b) $\Omega = 0.5$; and (c) $\Omega \to 1$.

control interval and the *degree of asynchronism* are illustrated in Figure 2. Note that the case of $\Omega \to 1$ may correspond to the situation where the delay is close to the limit of the tolerable range and the system must sample extremely fast in order to control the unstable plant. When the delay is small ($\Omega \to 0$), the system may sample at a correspondingly lower rate and still be stable. In both the cases, the stability depends on Θ rather than the delay or sampling rate alone. Related observation has been reported in [15] for feedback control over wireless communications links.

Here, unless otherwise stated, we simplify the *asynchronous action model* slightly by letting t_j's and p_j's be evenly spaced. Then $p_j - t_{j-1} = \Theta$, $p_j - t_j = \Theta \cdot \Omega$, and $t_j - t_{j-1} = \Theta \cdot (1 - \Omega)$ for any $j \geqslant 1$. The variable Θ can be viewed as the length of an atomic control cycle of the entire system. The control cycles of the individual virtual subsystems are constructed from these atomic cycles. In the case of $\Omega = 0$, the control cycles of the individual virtual subsystems must be multiples of Θ. When $\Omega \neq 0$, the lengths of the control cycles of the individual subsystems are $\Theta\Omega$ plus multiples of $\Theta(1 - \Omega)$. Note that with this simplification, we can still study the robustness of the system with respect to the fluctuation of communication bandwidth by examining whether our control techniques can tolerate the variations of Θ and Ω.

Assuming the above communication time scheme, we next examine the content and amount of the communication in the feedback loop. In the DFCB control system we are considering, the control law of each virtual subsystem can be viewed as a mapping

$$v_{j,i} = f_i(x_i(t_j)) : \mathbb{R} \to \mathscr{V}_i$$

where $|\mathscr{V}_i| = N_i$, and t_j is a sampling instant when x_i is sampled. Thus, $\lceil \log_2 N_i \rceil$ bits of data need to be communicated during the control cycle from t_{j-1} to p_j. Taking account of differences that may exist in the time constants of component subsystems in (8), we will study the effect of occasionally skipping the sampling of certain virtual subsystem components. Thus, the amount of communication the entire system needs at the jth control cycle is, in number of bits,

$$\sum_{i=1}^{n} L_{j,i}$$

where

$$L_{j,i} = \begin{cases} \lceil \log_2 N_i \rceil & \text{if } x_i \text{ is sampled at } t_j \\ 0 & \text{otherwise} \end{cases}$$

DOI: 10.1002/rnc

In DFCB control, the choice of control strategy (Θ, N_i's, etc.) are constrained by the data rates. In [9], we have introduced the *base data rate*

$$R_{\rm b} = 1/\delta$$

where δ is the total time required for processing (generating, transmitting and receiving, etc.) each one bit of data between the plant and controller. (Here, we assume δ is for round trip.) If $R_{\rm b}$ and $L_{j,i}$'s are given, then the control interval Θ of the overall system is

$$\Theta = \frac{\max_{j\geqslant 1} \sum_{i=1}^{n} L_{j,i}}{R_{\rm b}} \tag{9}$$

i.e. Θ, the atomic control cycle of the overall system, needs to be long enough so that the required communication can be completed at rate $R_{\rm b}$ even at the busiest time—when all subsystems require some attention. Conversely, given a performance criterion and a feasible control law, one can compute the tolerable range of Θ (see [9, 12]). Then, the required $R_{\rm b}$, denoted by $R_{\rm b}^*$, can be calculated from (9). In the present paper, we always refer to the *required base data rate* $R_{\rm b}^*$ as the slowest data rate that permits a bounded response from some set of initial conditions.

Definition 1 (Li and Baillieul [9])
A DFCB control system on $\mathbb{R}^n$ produces bounded response if for some bounded domain D with non-zero Lebesgue measure, any trajectory started in D remains in D for all time.

Remark 3.1
It is important to note that in the case of varying data rate, the existence of a code-based control law that produces bounded response as defined above is a necessary condition for the 'containability' defined in [7]. Hence, producing bounded response is a very basic notion of stability for studying control over capacity-varying feedback channels.

In addition to the *base data rate*, here we are also concerned with the *average data rate*, which can be defined as

$$R_{\rm a} = \lim_{J\to\infty} \frac{\sum_{j=1}^{J} (\sum_{i=1}^{n} L_{j,i})}{t_J - t_0} \tag{10}$$

and the *required average data rate* $R_{\rm a}^*$ that is also induced by the performance criterion of producing bounded response.

$R_{\rm a}$ is related to $R_{\rm b}$. For instance, when $\sum_{i=1}^{n} L_{j,i} \equiv L$ for all j's, $R_{\rm a} = R_{\rm b}/(1-\Omega)$. (Recall that Ω is the degree of asynchronism, which also reflects how much the adjacent control cycles overlap with each other.) Further, if $\Omega = 0$ (simultaneous sampling and actuation), $R_{\rm a} = R_{\rm b}$. On the other hand, when the sampling is scheduled and skips certain modes at certain times ($L_{i,j} = 0$ for some (i,j)'s), $R_{\rm a}$ can be lower than $R_{\rm b}$. We will explore such strategies in the sequel.

Extending the theoretical data-rate limit ([4, 5]) to our set-up, the tightest lower bound for the instantaneous and average data rates are

$$R_{\rm b}^* \geqslant R^* = \log_2 e \cdot \sum_{i=1}^{n} a_i, \quad R_{\rm a}^* \geqslant R^*/(1-\Omega) \tag{11}$$

for the closed-loop system to produce bounded response. Note that in the derivation of the data-rate inequality (1), it is commonly assumed that the control actions take the effect

DOI: 10.1002/rnc

instantaneously after sampling. The factor $(1-\Omega)$ in the bound of R_a^* reflects that when the control actions do not take effect instantaneously, the samples need to be taken at higher rates. Suppose the virtual control law for each (scalar) subsystem is binary (i.e. $N_i = 2$) and all subsystems are sampled at all sampling times, i.e. $L_{j,i} \equiv 1$. (We refer to this strategy as the *simple strategy*.) Then according to [9], $\Theta \leqslant (\log_2 e \cdot \max a_i)^{-1}$ must be satisfied (for bounded response). Also assume no asynchronism, the data-rate requirements are

$$R_a^* = R_b^* > \log_2 e \cdot n \max_{i=1,\dots,n} a_i \tag{12}$$

Clearly, this is greater than the theoretical minimum data rate of (11) and the reason is due to the lack of data-rate allocation among faster and slower modes. The remaining part of the paper will discuss the data-rate allocation strategies.

4. DISADVANTAGE OF ALLOCATING COMMUNICATION BANDWIDTH BY ALLOCATING QUANTIZATION LEVELS

The main contribution of the present paper is to propose a data-rate allocation strategy based on allocating time slots, which—together with the *virtual systems* approach—extends the robustness and efficiency of one-bit control to multidimensional control systems. Before discussing the time slot allocating strategy, an explanation of the advantage margin of the one-bit control over the control with larger alphabet but less frequent updates is in order.

4.1. Tolerance of asynchronism

An important advantage of the one-bit control lies in its tolerance of sampling-actuation asynchronism. Consider the DFCB control of a scalar system with normalized open-loop pole

$$\begin{aligned}
&\dot{x}(t) = x(t) + v(t)\\
&v(p_j \leqslant t < p_{j+1}) = f(x(t_j))\\
&p_0 = t_0 = 0, \quad p_{j+1} \geqslant p_j \geqslant t_j\\
&\sup_{j\geqslant 1}\{p_j - t_{j-1}\} = \Theta, \quad \sup_{j\geqslant 1}\{p_j - t_j\}/\Theta = \Omega, \quad j = 0, 1 \dots
\end{aligned} \tag{13}$$

where $\Theta \geqslant 0$, $0 \leqslant \Omega \leqslant 1$, and code-based control law

$$f(x) : \mathbb{R} \rightarrow \mathscr{V} \tag{14}$$

in which $\mathscr{V}$ is a finite control alphabet. Let $N = |\mathscr{V}| \in \{2, 3, \dots\}$. For this scalar system,

$$R_b = \frac{\log_2 N}{\Theta} \tag{15}$$

Suppose $f(\cdot)$ is given for each value of N. Then the required based data rate of the closed-loop system (for bounded response) is a function of the control alphabet length N and the sampling-actuation asynchronism Ω, hence can be denoted by $R_b^*(N, \Omega)$.

$R_b^*(N, \Omega)$ can be calculated numerically. The control law (with N being even)

$$\begin{aligned}
&f(x) = -b_l \quad \text{if } x \in [b_{l-1}, b_l)\\
&b_l = 2^{2l/N} - 1, \quad l = 1, \dots, N/2\\
&f(-x) = -f(x) \quad \text{for } x \neq 0
\end{aligned}$$

DOI: 10.1002/rnc

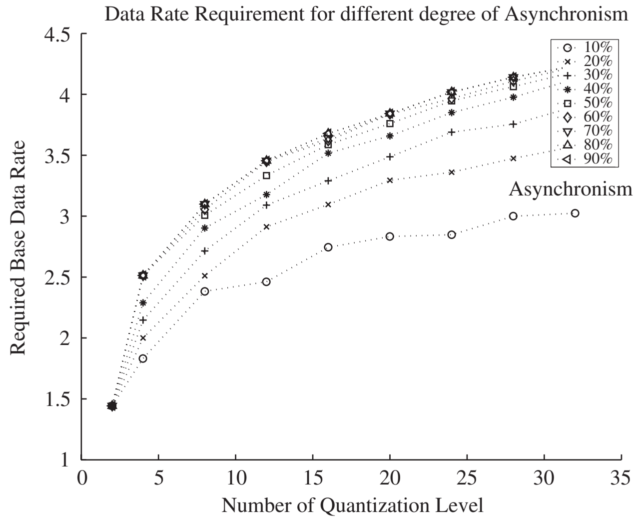

Figure 3. Required base data rates with various degree of asynchronism and length of control alphabet. The fact that the control alphabet must be coded by an integer number of bits is ignored.

was assumed in [12]. These control laws have been shown to require the least base data rates among *regular* control designs with given N (see [9]). For $N = 2$ (and only for $N = 2$) this control has $R_b^*(N, \Omega)$ equal to the theoretical bound, (1), given by the Data-Rate Theorem. The corresponding $R_b^*(N, \Omega)$ is shown in Figure 3.

More insight about the effect of asynchronism can be gained from the following theorems, which do not depend on the specific forms of the control laws.

Theorem 1

For the scalar DFCB control system (13), (14), for all $f(\cdot) \in F_N$, where F_N is the collection of code-based control laws with N distinct control actions, for all $\Omega > 0$, $R_b^*(N, \Omega) \to \infty$ as $N \to \infty$.

Proof

If the code-based control law $f(\cdot)$ is able to produce bounded response in a non-degenerate bounded interval D, then $\Delta = \sup|x + f(x)| > 0$ exists for $x \in D$. Note that $f(x)$ can only take a finite number of values. Suppose the sampling and control updating instants follow

$$\begin{aligned} p_0 &= t_0 = 0 \\ p_{j+1} &= t_j + \log\log N \\ t_{j+1} &\leqslant p_j - \log\log\log N, \quad j = 0, 1, \ldots \end{aligned} \tag{16}$$

where the base of the logarithm is natural. Then the following can be verified with tedious calculations: for N large enough, there exists $x_0 \in D$ and $\phi > 0$ such that $|x_0 + f(x_0)| > \phi\Delta$, and if $x(0) = x_0$, then, depending on the sign of $x_0 + f(x_0)$, either $x(t) + u(t) > \phi\Delta$, for all $t > 0$ or $x(t) + u(t) < -\phi\Delta$, for all $t > 0$, i.e. there exist initial states for which the state trajectory will be monotonically increasing or decreasing with non-diminishing speed, hence the trajectory cannot be contained in D. Thus, the system does not achieve *bounded response*.

DOI: 10.1002/rnc

On the other hand, (16) implies

$$\Omega \geqslant \frac{\log\log\log N}{\log\log N} \to 0, \quad R_{\rm b} = \frac{\log_2 N}{\log\log N} \to \infty$$

as $N \to \infty$. Since these values of Ω and $R_{\rm b}$ have been proved to be not stabilizing, then

$$\text{for all } \Omega > 0, \quad R_{\rm b}^* > \frac{\log_2 N}{\log\log N} \to \infty$$

as $N \to \infty$. □

4.2. *Tolerance of disturbance*

The next theorem quantifies the limitation of fine quantization from another perspective, namely, how the required data rate changes when there is disturbance. The result claims that the increase of required data rate from the theoretical limit in the presence of disturbances is minimized only when the single-bit control is applied. Intuitively, this is because the ratio between the magnitude of disturbance and the average size of the 'quantization cells' for the state feedback increases as the quantization becomes finer. In addition, here we derive the data-rate requirement in the presence of bounded disturbances, which will be useful in later sections. Here, we assume for simplicity that the sampling instants are evenly placed and there is no asynchronism ($\Omega = 0$), and write $R_{\rm b}^*(N)$ instead of $R_{\rm b}^*(N, \Omega)$.

Theorem 2

Suppose the plant dynamics in (13) has a disturbance term in addition:

$$\dot{x}(t) = x(t) + v(t) + w(t)$$

where $w(t)$ is an unknown but bounded disturbance. Consider the control problem of producing bounded response in the interval D of normalized length $\mu(D) = 2$, with $|w| \leqslant \kappa$ under the same normalization. Let F_N be the collection of code-based control laws with N distinct control actions. Then for all $f(\cdot) \in F_N$,

(i)
$$R_{\rm b}^*(N) \geqslant \log_2 e \cdot \frac{\log_2 N}{\log_2(1 + (N-1)/(N\kappa + 1))}$$

(ii)
$$R_{\rm b}^*(2) = \log_2 e \cdot \frac{1}{\log_2(1 + 1/(2\kappa + 1))}$$

with the one-bit control law

$$f(x) = \begin{cases} 1+\kappa, & x \geqslant 0 \\ -(1+\kappa), & x < 0 \end{cases}$$

and

(iii) for all $N > 2$ and all $f(\cdot) \in F_N$, $R_{\rm b}^*(N) > R_{\rm b}^*(2)$ when the one-bit control law is as given in (ii).

Proof

The proof of (i) follows the spirit of the Data-Rate Theorem. Suppose the range of $f(\cdot)$ is $\mathscr{V} = \{v_1, v_2, \ldots, v_N\}$ and let $\chi_l = \{x \in D | f(x) = v_l\}$. Then $\bigcup_{l=1}^N \chi_l = D$, hence for at least one χ_l,

DOI: 10.1002/rnc

the length $\mu(\chi_l) \geqslant 2/N$. Consider the initial states contained in such an interval χ_l and the first control cycle with time t running from 0 to Θ. The state transformation is

$$x(\Theta) = e^{\Theta} x(0) + (e^{\Theta} - 1) f(x(0)) + \int_0^{\Theta} e^{\Theta - \tau} w(\tau) \, d\tau \tag{17}$$

Let W denote the collection of all possible $w(t)$'s. Then the state transition (17) maps each $\chi_l \times W$ onto an interval χ_l^+ whose length is

$$\begin{aligned} \mu(\chi_l^+) &\geqslant \mu(\chi_l) \cdot e^{\Theta} + 2\kappa(e^{\Theta} - 1) \\ &\geqslant \frac{2}{N} e^{\Theta} + 2\kappa(e^{\Theta} - 1) \end{aligned} \tag{18}$$

(The first inequality in (18) becomes an equality if χ_l is connected, which is usually the case. The second inequality becomes an equality when all χ_l's have the same length, which is not uncommon either.) For bounded response in the sense explained previously, $\mu(\chi_l^+)$ must not exceed 2. This gives

$$e^{\Theta} \leqslant 1 + \frac{N-1}{N\kappa + 1} \tag{19}$$

(19) and (15) together give the bound in (i). The right-hand side of the inequality (i) is increasing as a function of N, (ii) may be verified by a direct calculation, and then (iii) follows. □

Section 2 has shown the decoupling of the multidimensional system (2) into the parallel interconnection of scalar subsystems (8) by means of the virtual systems approach. Thus, the data-rate requirement of (2) is related to those of the scalar systems in terms of the virtual systems. More precisely, let $R^*_{b,i}$ denote the required base data rate for a scalar system that has the same eigenvalue as the ith mode of the multidimensional system. Then the data-rate requirement of (2) is bounded from below by $\sum_{i=1}^{n} R^*_{b,i}$ and we will show in the sequel that this bound can be approached. So, by discussing the scalar systems, the above theorems and numerical results suggest that the potential of efficiently allocating feedback data rates by using large alphabets for faster modes is very limited; and it is worthwhile exploring strategies that are based on temporal allocation.

5. ALLOCATING COMMUNICATION BANDWIDTH BY ALLOCATING TIME SLOTS

Throughout this section, we assume that binary control is used for each virtual subsystem and the communication scheme follows the simplified asynchronous action model described at the beginning of Section 3. The reason why the *simple strategy* may require higher data rates than the theoretical minimum is that faster modes and slower modes have been given equal share of the bandwidth. There are a number of ways to address this problem.

Example 5.1

Assume $\Omega = 0$. Suppose $a_i = 1/z_i, \ i = 1, \ldots, n$ where z_i's are some integers that do not have any common factors. It seems fair that the ith virtual subsystem be sampled every z_i sampling

DOI: 10.1002/rnc

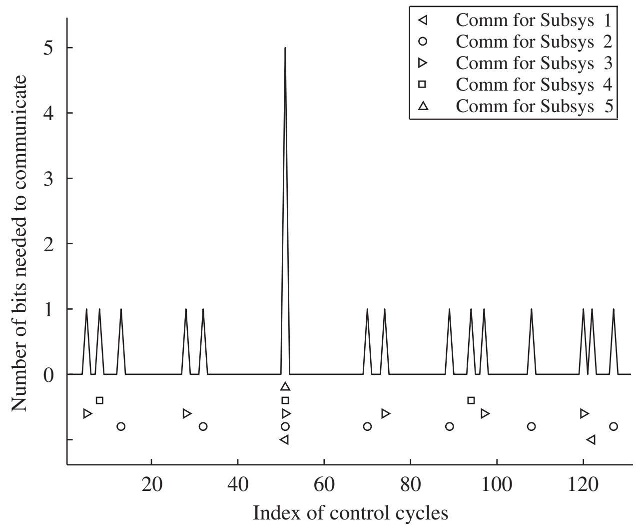

Figure 4. Data flow of example 5.1—A five-dimensional system is considered here with $a_1 = \frac{1}{71}$, $a_2 = \frac{1}{19}$, $a_3 = \frac{1}{23}$, $a_4 = \frac{1}{43}$, $a_5 = \frac{1}{87}$. Required R_b is $5 \log_2 e$ because of the data-flow peak. Required R_b with the *simple strategy* would be $\frac{5}{19} \log_2 e$.

instants. Then it sees a control interval of $z_i\Theta$ where Θ is the control interval on the shared communication link. The least upper bound of Θ for each subsystem to have bounded response is given by

$$z_i\Theta > (a_i \log_2 e)^{-1}$$

i.e. $\Theta < (\log_2 e)^{-1}$. The average data rate requirement for the entire system is then $R_a > \sum_{i=1}^{n} (1/z_i)/\Theta = \log_2 e \sum_{i=1}^{n} a_i$, which is exactly the theoretical minimum data rate. However, the required R_b in this case is $n \log_2 e$, which can be much larger than that of the simple strategy. Figure 4 illustrates the data flow under such a bandwidth allocation method. Note that the communication channel is transmitting very little data for most of the time except for some extremely busy times. Also note that because z_i's do not have any common factors, there are always some busy control cycles at which every subsystem needs update (the Euclidean Algorithm). Because this approach leaves a large amount of channel capacity unused for a large fraction of the time of operation, it is not ideal.

We shall next consider the bandwidth allocation techniques that will be generally referred to as *attention varying*. These techniques utilize novel *control magnitude protocols* and provide procedures for less attentive sampling of subsystems with larger time constant. Applying such procedures to the virtual systems described above provides ways of fairly allocating communication bandwidth in the feedback channel. Meanwhile, our techniques allow the same *base data rate* as the *simple strategy*. This is a clear advantage over the method in the last example.

For the system described by Equation (8), consider the problem of how to schedule the control of the virtual subsystems given nominal values of Θ and Ω of the common communication scheme. To simplify the discussion, we consider a control interval (of the

DOI: 10.1002/rnc

overall system) of normalized length

$$\Theta = (\log_2 e)^{-1} \tag{20}$$

With this temporal normalization, consider sampling the scalar subsystem (omitting the subscript)

$$\dot{x} = a(x + v + w) \tag{21}$$

where v is the control input and w is an unknown but bounded disturbance. Let the control objective be producing bounded response in the normalized range $(-1, 1)$ and under the same spatial normalization, assume $|w| \leqslant \kappa$ for all time. If neither asynchronism nor disturbance are present, i.e. $\Omega = 0$, $\kappa = 0$, we assume

$$0.5 < a < 1.0$$

Note that for this value of Θ, $a = 1$ is the largest possible value such that (21) will have a bounded response for some switching law of v. Note also that with the atomic control cycles of length $\Theta = (\log_2 e)^{-1}$, if $0 < a \leqslant 0.5$, (21) can be sampled only once every two or more atomic control cycles and still admit a feedback law which produces a bounded response. Thus, (21) can be identified as a scaled version of a component subsystem whose eigenvalue is less than the greatest eigenvalue of the overall system, but is not equal to the greatest eigenvalue divided by an integer. If the $R_{\rm a}$ related to the Θ in (20) is close to the theoretical limit for every such subsystem, then so is the $R_{\rm a}^*$ of the overall system.

In the presence of bounded disturbance of normalized magnitude κ, Theorem 2 has shown that the instantaneous data-rate requirement for a scalar system with eigenvalue a is

$$R_{\rm b}^* = \frac{a \cdot \log_2 e}{\log_2(1 + 1/(1 + 2\kappa))} \tag{22}$$

On the other hand, with asynchronism, the control interval for a component subsystem whose sampling skips $(k-1)$ atomic control cycles is $(k(1-\Omega) + \Omega)\Theta$. Accordingly, for the case with bounded disturbance and asynchronism, we assume

$$\frac{1}{2-\Omega} \log_2\left(1 + \frac{1}{1+2\kappa}\right) < a < \log_2\left(1 + \frac{1}{1+2\kappa}\right)$$

This can be verified from (22) to be the range of a corresponding to $(0.5, 1)$ in the no disturbance, no asynchronism case.

Let this scalar system be controlled by the binary law

$$v(t)|_{t \in [p_j, p_{j+1})} = \begin{cases} v(p_{j-1}) & \text{if sampling skipped} \\ \begin{cases} \sigma_j & x(t_j) < 0 \\ -\sigma_j & x(t_j) \geqslant 0 \end{cases} & \text{otherwise} \end{cases} \tag{23}$$

where σ_j is the magnitude of control for the jth control cycle (control action applied at p_j).

5.1. *An aggressive approach*

Here, we describe an approach that can reduce the $R_{\rm a}$ of the above subsystem arbitrarily close to the theoretical limit. However, this approach requires good knowledge of the lengths of the control intervals, hence is not robust to varying and uncertain feedback channel capacity.

DOI: 10.1002/rnc

Because of this drawback, we only describe this approach with the ideal assumption that $\Omega = 0$ and $\kappa = 0$. (Neither asynchronous control action nor disturbances are present.)

For any given $\varepsilon > 0$, there exist $m, k \in \mathbb{Z}^+$ such that $a \leqslant m/(m+k) < a + \varepsilon$. Assuming $x(t_0) \in (-1, 1)$, let the control system enter the following loop:

Stage 0: Set $\sigma_j = 2^{a-1}/2^a - 1$ and let subsystem (21) be sampled, and the control be updated, at the instant t_j. It can be verified that

$$|x(t_{j+1})| \leqslant 2^{a-1} = (2^a - 1)\sigma_j$$

(Note that this range for $x(t_{j+1})$ is only valid for the specified Θ. The range would become invalid if Θ were larger or smaller.) Increase j by one and finish the control cycle ending at the new t_j.

Stage 1: Set

$$\sigma_j = 2^{a-1}\sigma_{j-1}$$

and let subsystem (21) be sampled and the control be updated. Increase j by one and finish the control cycle ending at the new t_j.

Repeat this stage for a total of $m-1$ control cycles. Note that the bound of $x(t_j)$ shrinks by 2^{a-1} each time.

Stage 2: After repeating the cycles in Stage 1,

$$|x(t_j)| \leqslant 2^{m(a-1)}$$

Set $\sigma_j = 0$ and run the next k control cycles without sampling, after which j increase by k and

$$|x(t_j)| \leqslant 2^{m(a-1)} \cdot 2^{ka} \leqslant 1$$

The last inequality holds because $a \leqslant m/(m+k)$.

Return to Stage 0.

Remark 5.1

(1) The above procedure repeats itself every $m + k$ control cycles, in which one-bit samples are taken for m times. Thus the average data rate is

$$R_{\mathrm{a}} = \frac{m}{m+k}\Theta^{-1} = \frac{m}{m+k}\log_2 e$$

which is arbitrarily close to the theoretical limit $a \log_2 e$.

(2) However, the above procedure requires precise knowledge of Θ to operate. Although the procedure can be modified to accommodate a range of values for Θ, fairly good knowledge of Θ is still needed for effective reduction of the average data rate. This somewhat limits its use in cases where the communication channel capacity is uncertain.

5.2. *A robust approach*

The rest of this paper will focus on the *robust attention-varying* technique, which is described below. Let the control system enter the following loop:

Stage 0: Set $\sigma_j = 1 + \kappa$ and let subsystem (21) be sampled at every common sampling instant t_j. If $x(t_{j*})$ is the first sample that indicates the state trajectory has crossed the origin,

DOI: 10.1002/rnc

then

$$|x(p_{j^*})| \leqslant (2^a - 1)(1 + 2\kappa)$$

(Note that if Θ takes the nominal value (20), then the above range is tight for $x(p_{j^*})$. If Θ takes a smaller value, the range is not tight, but still valid. This is a crucial feature that ensures the robustness of our procedure with regard to varying Θ, hence varying data rate.) Go to the next step upon getting such a sample.

Stage 1: Set

$$\sigma_{j^*} = (2^a - 1)(\sigma_{j^*-1} + \kappa) + \kappa$$

Maintain this value for σ_j until or unless the algorithm returns to this step.

Stage 2: Find the largest integer k such that

$$(2^{a(k(1-\Omega)+\Omega)} - 1)(\sigma_{j^*} + \kappa) < 1$$

Stage 3: If $k \geqslant 2$, set the sampling frequency to once every k atomic control cycles. (i.e. the control cycle for this subsystem becomes $k\Theta(1 - \Omega) + \Theta\Omega$.) Wait until the state trajectory crosses the origin again. Then return to Stage 0.

Stage 4: If $k < 2$, wait until the state trajectory crosses the origin, then let j^* be the index of the first sample that indicates this crossing and go to Stage 1.

The block diagram of the above procedure is shown in Figure 5. It is easy to verify that bounded response is ensured.

Remark 5.2

(1) Above is the technique we propose for allocating communication among the decoupled virtual subsystems. We will show that it reduces the *average data-rate* requirements. Note that when the *average data-rate* requirement for each individual virtual subsystem is close to the theoretical minimum, so is that of the overall system.

(2) This technique does not depend on precise knowledge of Θ. When the available data rate increases and Θ becomes smaller, the state trajectory will not only remain bounded, but also stay closer to the origin. With infinite data rate, the state will converge to 0.

(3) Moreover, although the knowledge of Ω is helpful, it is not necessary. Ω can be replaced by any of its lower bounds, or one can simply assume $\Omega = 0$ to apply the procedure, and still get bounded response and reduced average data rate requirement.

(4) The effect of Stage 1 in our algorithm is to reduce the size of the attracting set for motions of the controlled system. Techniques to accomplish similar objectives have been reported in [6, 9, 10].

(5) In addition, the present technique can be further synthesized with the 'side-channel' technique in [9] to achieve asymptotic stability.

(6) When

$$\kappa \geqslant \frac{1 - 2^{a-1}}{2^{a(2-\Omega)} - 1}$$

our procedure operates in a trivial way. Specifically, the updates in Stage 1 approach a limit such that $k \geqslant 2$ is never satisfied and Stage 3 is never reached. The average

DOI: 10.1002/rnc

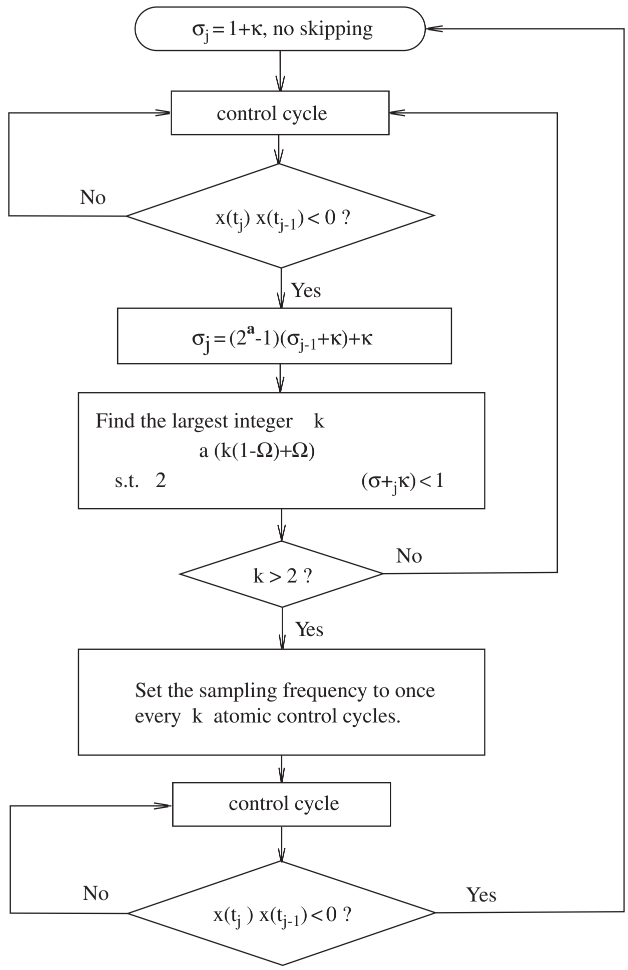

Figure 5. Block diagram of the robust attention varying data-rate allocation technique.

data-rate requirement is not reduced then. To illustrate the magnitude of disturbance that can result in this degradation, consider $a = 0.9$, $\Omega = 0.2$, the above inequality becomes $\kappa \geqslant 0.0323$. Also consider $a = 0.6$, $\Omega = 0.1$, the inequality becomes $\kappa \geqslant 0.2011$.

Next, we calculate the data-rate requirements. First, consider the *base data-rate*. If we set the nominal Θ to $(a_{\max} \log_2 e)^{-1}$ where $a_{\max}$ is the largest eigenvalue of the system, then the *base data-rate* requirement becomes no greater than that of the *simple strategy*, which is acceptable. Further reducing R_b seems difficult in general but is possible for specific systems (sets of eigenvalues), e.g. consider a four dimensional system with $a_1 = 1$ and a_2, a_3, a_4 smaller than but roughly equal to $\frac{1}{3}$. In terms of producing bounded response with coarsely quantized control, let the atomic control cycle length be $\Theta = 1/\log_2 e$, the controller only needs to send a two-bit command each time. One bit will be dedicated to the fastest mode a_1 and the three slower modes

DOI: 10.1002/rnc

take turns using the other bit. Then both the *base data-rate* and the *average data-rate* requirements are close to the theoretical minimum.

In the rest of this section, we calculate the *average data rate* under this technique. The closed-form calculation seems hard for arbitrary sets of parameters. But some particular values of a do allow closed-form calculations.

First, suppose $\Theta = (\log_2 e)^{-1}$, $\Omega = 0$, $\kappa = 0$ and a satisfies

$$(2^{2a} - 1)(2^a - 1) = 1 \tag{24}$$

Assume the initial state $x(t_0)$ is uniformly distributed in $(-1, 0)$. Then the sampling interval of this subsystem is switched from Θ to 2Θ whenever the state trajectory crosses the origin in the positive direction (negative to positive), and it is switched back when a crossing happens in the opposite direction. The state trajectory is contained in $(-1, 2^a - 1)$ (see the state-transition graph in Figure 6). To calculate the average data rate, we need to examine how much time the system spends with sampling interval Θ, and how much with 2Θ.

Call the state transition each between two adjacent samples in the sampling sequence a *step* and call the collection of steps between the instants when σ is reset to 1 a *round*. Index the *rounds* with l. Let the number of steps with sampling interval Θ in the lth round be Q_l and that with 2Θ be S_l. Now consider the first round. $Q_1 = 1, 2, \ldots$ is a discrete-valued random variable.

$$\begin{aligned} P(Q_1 = q) &= P(2^{-qa} - 1 \leqslant x(t_0) < 2^{-(q-1)a} - 1) \\ &= 2^{-qa}(2^a - 1) \end{aligned} \tag{25}$$

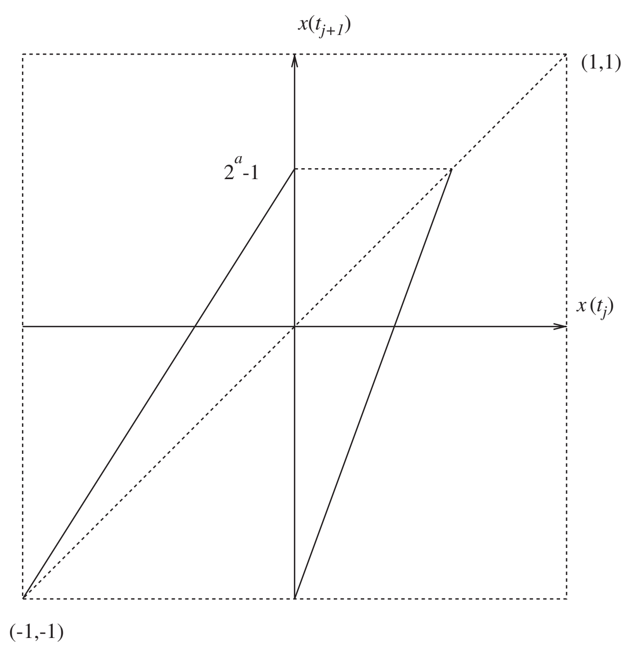

Figure 6. State transition for the special case.

The steps in the second half of round 1, S_1, is also a discrete-valued random variable taking value in all positive integers

$$P(S_1 = s, Q_1 = q) = P\left(1 - 2^{-2(s-1)a} \leqslant \frac{x(t_{Q_1})}{2^a - 1} < 1 - 2^{-2sa},\ Q_1 = q\right)$$

$$= P(1 - 2^{-2(s-1)a}) \leqslant \frac{(x(t_0) + 1)2^{qa} - 1}{(2^a - 1)} < 1 - 2^{-2sa}$$

$$= 2^{-qa} \cdot 2^{-2sa} \cdot (2^{2a} - 1)(2^a - 1). \tag{26}$$

From (25) and (26),

$$P(S_1 = s | Q_1 = q) = 2^{-2sa}(2^{2a} - 1) \tag{27}$$

So, S_1 is independent of the outcome of Q_1 and

$$P(S_1 = s) = 2^{-2sa}(2^{2a} - 1) \tag{28}$$

Similarly, the outcomes of Q_l's and S_l's in different rounds are independent of each other. Q_l's and S_l's are also i.i.d., respectively. (Although all the randomness comes from the initial condition.) Note that we do not have this nice property for an arbitrary a. Then, the *average data rate* for this subsystem is

$$R_{\mathrm{a}} \to \frac{E[Q] + E[S]}{E[Q] + 2E[s]} \log_2 e \quad \text{w.p.1.} \tag{29}$$

It is easy to calculate that

$$E[Q] = \frac{2^a}{2^a - 1} \quad \text{and} \quad E[S] = \frac{2^{2a}}{2^{2a} - 1} \tag{30}$$

To satisfy (24), $a \approx 0.6942$. Then from (29) and (30), $R_{\mathrm{a,sub}} \to 0.7340 \cdot \log_2 e$ with probability 1. The *average data rate* here exceeds the theoretical minimum by only 5.7%.

Above is the simplest case where the *average data rate* can be calculated in closed form. In fact, similar calculations hold for all a's that satisfy

$$(2^{ka} - 1)(2^a - 1)^m = 1 - 2^{-qa} \tag{31}$$

for some integers $k \in \{2, 3, \ldots\}$, $m \in \{1, 2, \ldots\}$, $q \in \{1, 2, \ldots\}$. These are the cases in which Stages 1, 2 and 4 of the procedure (reducing the size of the attracting set) are repeated for m times before Stage 3 is reached; and in Stage 3, only one sample is taken in every k atomic control cycles. Call the steps between the instants when $x(t)$ crosses the origin a subround. Then q is the maximum number of steps in the rounds immediately after σ is reset to 1. Note that all the other subrounds may contain infinite number of steps. The average data rates in these cases are

$$R_{\mathrm{a}} \to \frac{\sum_{r=1}^{m} E[Q^{(r)}] + E[S]}{\sum_{r=1}^{m} E[Q^{(r)}] + k \cdot E[S]} \log_2 e \quad \text{w.p.1.} \tag{32}$$

DOI: 10.1002/rnc

where

$$E[Q^{(1)}] = \sum_{i=1}^{q} i \cdot 2^{-ia} \frac{2^a - 1}{1 - 2^{-qa}}$$

$$E[Q^{(r)}] = \frac{2^a}{2^a - 1}, \quad r = 2, \ldots, m$$

and

$$E[S] = \frac{2^{ka}}{2^{ka} - 1} \tag{33}$$

6. SIMULATIONS

This section shows simulation results for cases that do not allow explicit calculations like those shown in the last section. First, in Figure 7, the results (dotted line for simulation results and diamonds for calculated results) are plotted together with the theoretical minimum data rates. The figure shows that the required *average data rates* with our technique are very close to the theoretical minimum.

Next, in Figure 8, the average data rate under the robust attention varying technique is plotted in dotted line for the cases in which disturbance and sampling-actuation asynchronism are present. Plotted in solid line, the tightest lower bound for required average data rate corresponding to this case is

$$R_{\rm a}^{*} \geqslant \frac{a \cdot \log_2 e}{\log_2(1 + 1/(1 + 2\kappa))} \cdot \frac{1}{1 - \Omega} \tag{34}$$

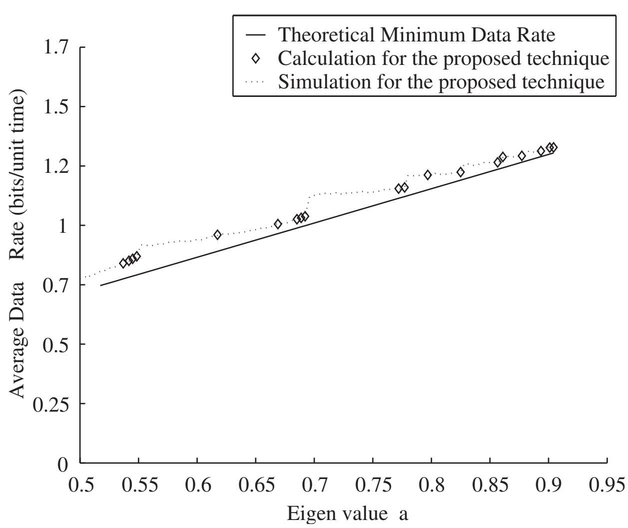

Figure 7. Average data rate with robust attention varying.

DOI: 10.1002/rnc

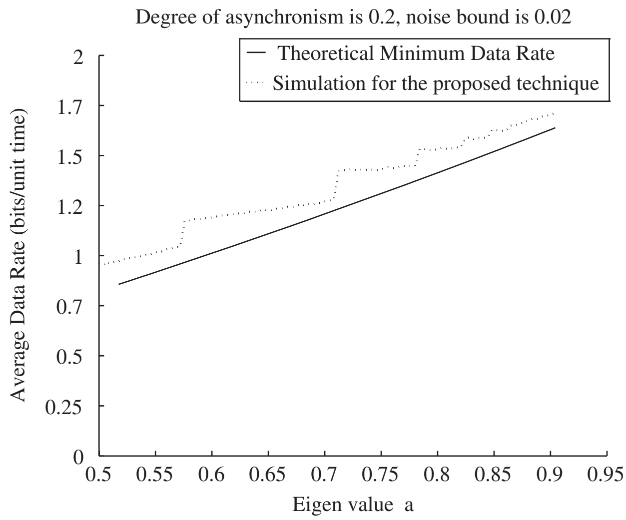

Figure 8. Average data rate with the robust attention varying, the case with disturbance and asynchronism. Assume $\kappa = 0.02$ and $\Omega = 0.2$.

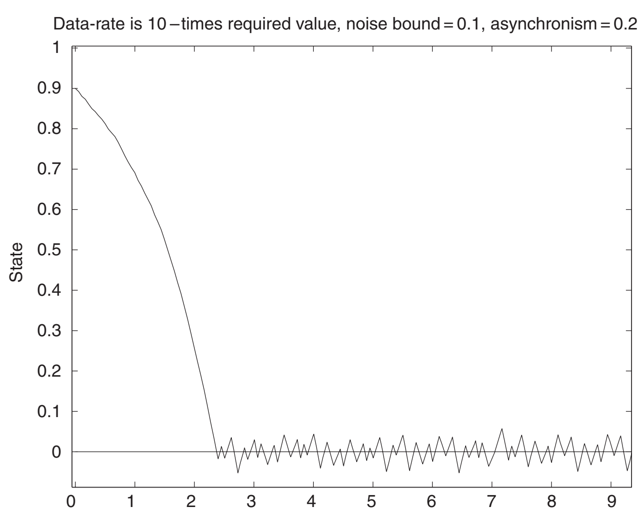

Figure 9. State trajectory with high data rate. The data rate is 10 times higher than the nominal value. $\kappa = 0.1$, $\Omega = 0.2$.

Finally, Figure 9 plots a state trajectory of the (sub)system for the case where the actual data rate is much higher than the nominal value. It shows that the state is well contained around the origin by the control with the high data rate.

DOI: 10.1002/rnc

7. CONCLUSION

In [8], it was noted that quantized implementations of classical feedback designs typically require data rates in the feedback channel which are significantly higher than the theoretically minimum. In the present paper, we have proposed a novel coding approach utilizing a form of 'distributed intelligence' together with the concept of a *virtual system*. Further, we discussed feedback communication scheduling techniques aimed at efficiently allocating the available communication bandwidth among different modes. In particular, the *robust attention-varying* technique that we proposed, combined with the *virtual systems* approach for feedback and control coding, allows operation of the closed-loop system that is robust with respect to variations in the feedback channel capacity, and also provides satisfactory performance at data rates near the theoretical minimum.

In this paper, we have restricted our discussion to systems with distinct real eigenvalues. Cases with Jordan blocks of sizes larger than one are more complicated, see [13]. However, the advantages of using small control alphabets, as seen in the scalar Jordan blocks cases, are still worth considering. In addition, our approach of control decoupling, and that of data-rate allocation by identifying opportunities of removing attention from relatively slow modes of the system, are still applicable in a qualitative sense.

There are many interesting open questions regarding source coding for feedback control over data-rate constrained communications channels. While coding techniques can be proposed which actually provide for bounded system response when operation is arbitrarily near the theoretical minimum, those of which we are aware do not have the robustness to variations in channel capacity that we have discussed above. Current research is aimed at understanding the tradeoffs involving such robustness, ability to operate at extremely low data rates, and other quality of performance issues.

ACKNOWLEDGEMENTS

This work was supported by ODDR&E MURI01 Program Grant Number DAAD19-01-1-0465 to Boston University and the Center for Networked Communicating Control Systems, and by the National Science Foundation ITR Program Grant Number DMI-0330171.

REFERENCES

1. Baillieul J. Feedback designs for controlling device arrays with communication channel bandwidth constraints. *ARO Workshop on Smart Structures*, Pennsylvania State University, Pennsylvania, August 16–18.
2. Nair GN, Evans RJ. Stabilization with data-rate-limited feedback: tightest attainable bounds. *Systems and Control Letters* 2000; **41**(1):49–56.
3. Tatikonda S. Control under communication constraints. *Ph.D. Thesis*, MIT, Cambridge, MA, September 2000.
4. Nair GN, Evans RJ. Exponential stability of finite-dimensional linear systems with limited data rates. *Automatica* 2002; **39**:585–593.
5. Tatikonda S, Mitter SK. Control under communication constraints. *IEEE Transactions on Automatic Control* 2001; **49**(7):1056–1068.
6. Brockett RW, Liberzon D. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control* 2000; **45**(7):1279–1289.
7. Wong WS, Brockett RW. Systems with finite communication bandwidth constraints II: stabilization with limited information feedback. *IEEE Transactions on Automatic Control* 1999; **44**:1049–1053.
8. Baillieul J. Feedback coding for information-based control: operating near the data-rate limit. *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, December 2002; 3229–3236.

9. Li K, Baillieul J. Robust quantization for digital finite communication bandwidth (DFCB) control. *IEEE Transactions on Automatic Control*, Special Issue on Networked Control Systems 2004; **49**:1573–1584.
10. Fagnani F, Zampieri S. Quantized stabilization of linear systems: complexity versus performance. *IEEE Transactions on Automatic Control*, Special Issue on Networked Control Systems 2004; **49**:1534–1548.
11. Li K, Baillieul J. Robust quantization and coding for control of multidimensional linear systems under data rate constraints. *Proceedings of IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004; 1920–1925.
12. Li K, Baillieul J. 2004. Robust quantization for control with time-varying feedback communication constraints. *Proceedings of the 5th Asian Countries Conference*, Melbourne, Australia, July 2004; 1906–1914.
13. Li K. Robust and efficient feedback coding for communication based control: operating under data-rate constraints. *Ph.D. Thesis*, Boston University, Boston, May 2005.
14. Li K, Baillieul J. Robust quantization for digital finite communication bandwidth (DFCB) control. *Proceedings of 42nd IEEE Conference on Decision and Control*, Maui, HI, December 2003; 3130–3135.
15. Suri A, Baillieul J, Raghunathan DV. Control using feedback over wireless ethernet and bluetooth. In *Handbook of Networked and Embedded Control Systems*, Hristu-Varsakelis D, Levine WS (eds). Birkhäuser: Boston, 2005; 677–697.

DOI: 10.1002/rnc