ENG ME/EC 501:

First Hour Quiz (Hint: Do the easy ones first.)

30 pts 1. Compute $e^{At}$ where $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$.

2. Consider the system $\dot{x}(t) = Ax(t)$, $x(t_0) = x_0$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -5 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

20 pts a) Let $tr(A)$ denote the trace of $A$, and let $\Phi(t, t_0)$ be the corresponding transition matrix. Show that:

$$\det[\Phi(t, t_0)] = e^{\int_{t_0}^{t} tr(A) dt}$$

5 pts b) Is the system asymptotically stable or unstable? (Explain your answer)

30 pts 3. Let $g(s) = \frac{s^2+6s+8}{s^3+9s^2+23s+15}$.

(a) Write down the standard controllable realization.

(b) Write down the standard observable realization.

15 pts 4. (a) The characteristic polynomial of an $n \times n$ matrix $A$ has the form

$$p(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0.$$ 

In particular, the characteristic polynomial of a $5 \times 5$ matrix has the form

$$p(s) = s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0.$$
For

\[
A = \begin{pmatrix}
2 & -2 & -4 & 0 & 0 \\
-1 & 3 & 4 & 0 & 0 \\
1 & -2 & -3 & 0 & 0 \\
a & b & c & 1 & 0 \\
x & y & z & 0 & 1
\end{pmatrix}
\]

with \(a, b, c, x, y, z\) unspecified, find \(a_4\).
Exam Solutions

1. \( A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \). Eigenvalue is 3 with multiplicity 2.

\[
\det[sI - A] = (s - 2)(s - 4) + 1 = s^2 - 6s + 9 = (s - 3)^2.
\]

Compute the kernel of \( A - 3I \):

\[
\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}
\]

The null space is spanned by \( \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Next, note that

\[
(A - 3I)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Hence the null space is all of \( \mathbb{R}^2 \). There are many (\( \infty \)'ly many) choices for the second generalized eigenvector \( \vec{u} \), but it must have the property that

\( (A - 3I) \vec{u} = \vec{v} \). A simple choice is \( \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Consider the coordinate transformation defined by these two generalized eigenvectors:

\[
U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

Note that \( U^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \). Then

\[
U^{-1}AU = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = J_A
\]

From this we compute \( e^{At} = U e^{Jt} U^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} =
\]

\[
\begin{pmatrix} e^{3t}(-1 + t) & e^{3t}t \\ -e^{3t}t & e^{3t}(1 + t) \end{pmatrix}
\]

2. a) There is a very elegant solution to this problem on page 28 of the Brockett text. Here are some others (not quite as general).

Suppose \( A \) is a constant \( n \times n \) matrix. Recall that for any two \( n \times n \) matrices \( M \) and \( N \) that \( \text{tr}[MN] = \text{tr}[NM] \). Also \( \text{det}[MN] = \text{det}[M] \text{det}[N] \). Now let \( U \) be a nonsingular matrix that by conjugation puts \( A \) into its Jordan normal form:

\[
U^{-1}AU = J_A
\]

Then \( \text{det}[e^{At}] = \text{det} U \text{det}[e^{Jt}] \text{det} U^{-1} = \text{det}[e^{Jt}] \). By means of a direct calculation, for a matrix in JNF, \( \text{det}[e^{Jt}] = e^{\text{tr}[Jt]} \). The result follows from noting that \( e^{\text{tr}[At]} = e^{\text{tr}[Jt]} \). A way to do it using the explicit form of the matrix in the problem is on the next page.
\( \Phi(t, s) = e^{A(t-s)} \). There exists some \( P \), \( \det P \neq 0 \), such that \( \tilde{A} = P A P^{-1} \) is in Jordan Canonical form.

\[ \Rightarrow \Phi(t, s) = P^{-1} \Phi(t, 0) P. \] (Don't need to compute this \( P \)).

What are the eigenvalues of \( \tilde{A} \)?

\[
\det (sI - \tilde{A}) = (s-1) (s+1) (s+1) (s+1) (s+2) = (s-1) (s+3) (s+5) + 2.1
\]

\[
= (s-1)(s+1)(s+1)(s+1)(s+2)
\]

\[
\Rightarrow \tilde{A} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\Rightarrow e = \begin{bmatrix}
-e^{-e(t-t_0)} \\
e^{-e(t-t_0)} \\
e^{-e(t-t_0)} \\
e^{-e(t-t_0)}
\end{bmatrix}
\]

Now \( \det \Phi(t, s) = \det P \det \Phi(t, 0) \det P^{-1} = \det \Phi(t, s) \)

\[
= e^{-3(e(t-t_0))} \int_t^{t+1} e^{A(t-t)}
\]

2 b) The most obvious eigenvalue of the matrix is 1, and this being in the right half plane means that the system is unstable.
3(a).

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ C = (8, 6, 1) \]

\[ \det \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \det \begin{pmatrix} 8 & 6 & 1 \\ -15 & -15 & 3 \\ 45 & 54 & 12 \end{pmatrix} = -9 \]

Thus, the system is observable.

3(b). \[ S^3 + 9S^2 + 23S + 15 \]
\[ \frac{S^{-1} - 3S^{-2} + 12S^{-3} + \ldots}{\sqrt{S^2 + 6S + 8}} \]
\[ \frac{S^2 + 9S + 23 + 15S^{-1}}{\ldots} \]
\[ -3S - 15 - 15S^{-1} \]
\[ -3S - 27 - 69S^{-1} - 45S^{-2} \]
\[ 12 + 54S^{-1} + 45S^{-2} \]
\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -3 \\ 12 \end{pmatrix} \]

\[ C = (1, 0, 0) \]

\[ (B, AB, A^2B) = \begin{pmatrix} 1 & -3 & 12 \\ -3 & 12 & -54 \\ 12 & -54 & 255 \end{pmatrix} \]

\[ \det (B, AB, A^2B) = 9 \quad \therefore \text{CONTROLLABLE.} \]

4. Either using the algorithm to compute the resolvent, or by means of a direct calculation, we find that \( a_4 = -\text{tr} [A] = -4. \)