Matrix Exponentials

Motivation for today’s lecture: In robotics we frequently encounter the rotation of some rigid body about some given axis by some amount. Consider the rotation of some link about the z-axis as shown below:

Figure 1: Link Rotating

Suppose $\theta(t)$ is the rotation angle in radians and suppose the link rotates at a constant, unit velocity, so that: $\theta(t) = t$, $\dot{\theta}(t) = 1$. As a result, the coordinate of the link tip at time $t = 0$ and $t = 1$ can be respectively expressed as:

$$ p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} $$

The velocity of the tip is therefore given by:

$$ \dot{p}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} $$

Which can be be expressed as:
\[ \dot{p}(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} p(t) \]

\[ \dot{p}(t) = Ap(t) \]

This is a differential equation in state-space form with \( A \) being a constant matrix.

In general, if the axis of rotation is given by some unit vector:

\[ w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \]

one can show that:

\[ \dot{p}(t) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} p(t) = w \times p(t) \]

Again a differential equation in state-space form:

\[ \dot{x}(t) = Ax(t) \quad x(0) = x_0 \]

If we know \( p(0) \) and \( w \), we can compute \( p(t) \) by solving this differential equation.

**Theorem 0:** Let \( A \) be an \( n \times n \) matrix with constant entries and let the sequence of matrices be defined recursively as follows:

\[ M_0 = I \]

\[ k \geq 1, \quad M_k(t, 0) = I + \int_0^t A M_{k-1}(\sigma, 0) d\sigma \]

The the sequence of matrices \( M_0, M_1, M_2, \ldots \) converges uniformly on any time interval \( 0 \leq t \leq t_1 \). Moreover, if the limit is defined as \( \Phi(t, 0) \) (i.e., \( \Phi(t, 0) = \lim_{k \to \infty} M_k(t, 0) \)) then for \( 0 \leq t \leq t_1 \):

\[ \frac{d\Phi(t, 0)}{dt} = A\Phi(t, 0), \quad \Phi(0, 0) = I \]

and the solution of \( \dot{x}(t) = Ax(t) \), \( x(0) = x_0 \), is given by \( x(t) = \Phi(t, 0)x_0 \).

Before we prove this theorem it will be helpful if we recall some basic facts/definitions/statements from the theory of convergence of sequences of functions.

**D/F/S 1:** Let \( f_i(t) \) be a sequence of scalar real valued functions defined on the interval \( T : 0 \leq t \leq t_1 \). A sequence of functions \( f_1(t), f_2(t), \ldots \) define on \( T \), is said to converge (pointwise) to some function \( f(t) \) if
\[
f(t) = \lim_{n \to \infty} f_n(t) \quad \text{for every } t \in T
\]

**D/F/S 2:** A sequence is said to converge **uniformly** to \(f(t)\) on \(T\), if for every \(\epsilon > 0\), there exists an \(N\) (depending on \(\epsilon\) not \(t\)) such that for \(n > N\)

\[
|f_n(t) - f(t)| < \epsilon \quad \text{for every } t \in T
\]

**D/F/S 3:** A series of functions \(f_1(t) + f_2(t) + f_3(t) + \cdots\) defined on \(T\) is said to converge to a function \(f(t)\) if the sequence of partial sums \(\{s_i(t)\}\) converges to \(f(t)\) where:

\[
\begin{align*}
s_1(t) &= f_1(t) \\
s_2(t) &= f_1(t) + f_2(t) \\
\vdots \\
s_2(t) &= f_1(t) + f_2(t) + \cdots + f_n(t)
\end{align*}
\]

**D/F/S 4:** The series \(f_1(t) + f_2(t) + f_3(t) + \cdots\) converges uniformly to \(f(t)\) on \(T\), if the sequence \(\{s_i(t)\}\) converges uniformly to \(f(t)\) on \(T\).

**D/F/S 5:** Theorem (Weirstrass M-test). Let \(\{K_n\}\) be a sequence of non-negative numbers such that \(0 \leq |f_n(t)| \leq K_n\) for \(n = 1, 2, 3, \cdots\) and every \(t \in T\). Then the series \(f_1(t) + f_2(t) + f_3(t) + \cdots\) converges uniformly to \(f(t)\) on \(T\) if \(K_1 + K_2 + K_3 + \cdots\) converges.

**D/F/S 6:** Let \(A, B, A_1, A_2, \cdots, A_k\) be \(n \times n\) matrices with constant entries. Denote the \(i, j\) element of some matrix \(A\) as \(E_{i,j}(A)\). Let \(\alpha = \max_{i,j} |E_{i,j}(A)|, \quad \beta = \max_{i,j} |E_{i,j}(B)|, \quad \alpha_\ell = \max_{i,j} |E_{i,j}(A_\ell)|\). Then

\[
\left| E_{i,j}(AB) \right| \leq n\alpha\beta
\]

With \(A = (a_{i,j}), \quad B = (b_{i,j})\) then:

\[
\left| a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} \right| \leq |a_{i,1}b_{1,j}| + |a_{i,2}b_{2,j}| + \cdots + |a_{i,n}b_{n,j}|
\]

\[
\leq |a_{1,1}|b_{1,j}| + |a_{1,2}|b_{2,j}| + \cdots + |a_{1,n}|b_{n,j}|
\]

\[
\leq \alpha\beta + \alpha\beta + \cdots + \alpha\beta
\]

\[
\leq n\alpha\beta
\]

In general (via a proof by induction) one can show:

\[
\left| E_{i,j}(A_1A_2 \cdots A_k) \right| \leq n^{k-1}\alpha_1\alpha_2 \cdots \alpha_k
\]

**proof of the Theorem**
Step 1: Obtain expressions for the sequence of matrices $M_0(t, 0), M_1(t, 0), M_2(t, 0), \ldots$

\[
M_0(t, 0) = I
\]
\[
M_1(t, 0) = I + \int_{\sigma=0}^{t} A d\sigma = I + A \int_{\sigma=0}^{t} 1 d\sigma = I + At
\]
\[
M_2(t, 0) = I + \int_{\sigma=0}^{t} AM_1(\sigma, 0) d\sigma = I + \int_{\sigma=0}^{t} A(I + A\sigma) d\sigma = I + \int_{\sigma=0}^{t} A + A^2 \sigma d\sigma = I + At + \frac{1}{2} A^2 t^2
\]
\[
M_3(t, 0) = I + \int_{\sigma=0}^{t} AM_2(\sigma, 0) d\sigma = I + \int_{\sigma=0}^{t} A(I + A\sigma + \frac{1}{2} A^2 \sigma) d\sigma = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{2 \cdot 3} A^3 t^3
\]

Continuing in this manner we can write for $k \geq 1$:

\[
M_k(t, 0) = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{2 \cdot 3} A^3 t^3 + \cdots + \frac{1}{k!} A^k t^k
\]

Step 2: $M_k(t, 0)$ is a sum of $n \times n$ matrices and the $(i, j)$ element of its $k+1$ term can be bounded as follows for any $t$ in $0 \leq t \leq t_1$ (employing D/F/S 6):

\[
|E_{i,j}(\frac{1}{k!} A^k t^k)| \leq \frac{1}{k!} t^k |E_{i,j}(A^k)|
\]
\[
\leq \frac{1}{k!} t^k n^{k-1} \alpha^k
\]
\[
\leq \frac{1}{k!} t^k n^{k-1} \alpha^k
\]

So each element of $\frac{1}{k!} A^k t^k$ is bounded from above by the constant $\frac{1}{k!} t^k n^{k-1} \alpha^k$.

Step 3: This allows us to employ the Wierstrass M-test. Consider the series $\{K_k\}$:

\[
1 + \alpha t_1 + \frac{n(\alpha t_1)^2}{2!} + \frac{n^2(\alpha t_1)^3}{3!} + \cdots = 1 + \frac{1}{n} (e^{\alpha t_1} - 1)
\]

As we see the series of constants converges which implies that the sequence $\{M_k(t, 0)\}$ converges uniformly on $T$. We call this limit the “transition matrix.”
\[ \Phi(t, 0) = \lim_{k \to \infty} M_k(t, 0) = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots \]

It is a special case of the Peano-Baker series (A here a constant matrix), and we also denote it as:

\[ e^{At} = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots \]

and define it as the “matrix exponential.” Now, differentiating with respect to time term by term we obtain:

\[ \frac{d e^{At}}{dt} = 0 + A + A^2 t + \frac{1}{2} A^3 t^2 + \frac{1}{3!} A^4 t^3 + \cdots \]
\[ = A(I + At + \frac{1}{2} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots) \]
\[ = Ae^{At} \]

and where \( e^{A \cdot 0} = \Phi(0, 0) = I \).

Furthermore,

\[ \frac{d e^{At} x_0}{dt} = Ae^{At} x_0 \]
\[ (\star) \quad \dot{x}(t) = Ax(t), \quad x(0) = x_0 \]

Therefore, \( x(t) = e^{At} x_0 \) is the solution of (\( \star \)) above. We also can prove (not here) that this solution is unique.

**Note:** Since this is a “time-invariant” differential equation (\( A \) is constant), if the initial time is not 0 but rather \( t_0 \), the solution to (\( \star \)) is given by:

\[ x(t) = \Phi(t, t_0) x_0 = e^{A(t-t_0)} x_0 \]

**An Important Property of** \( \Phi(t, t_0) \), for arbitrary \( t_0, t_1, t \).

\[ \Phi(t, t_0) = \Phi(t, t_1) \Phi(t_1, t_0) \]

**Proof:** The unique solution of:

\[ (\star) \quad \dot{x}(t) = Ax(t), \quad x(t_0) = x_0 \]

is given by \( \bar{x}(t) = \Phi(t, t_0) x_0 \). Suppose that at time \( t_1, \bar{x}(t_1) = x_1 \). Consider again equation (\( \star \)) and develop its solution for initial condition \( x_1 \) at time \( t_1 \). In particular, we can write: \( \bar{x}(t) = \Phi(t, t_1) x_1 \). Since the solution to
(⋆) is unique (both solutions pass from $x_1$ at time $t_1$) we must have for all $t, t_1, t_0$ and all $x_0$ that:

$$\tilde{x}(t) = \tilde{x}(t)$$

This implies:

$$\Phi(t, t_0)x_0 = \Phi(t, t_1)x_1$$
$$\Phi(t, t_0)x_0 = \Phi(t, t_1)\Phi(t_1, t_0)x_0$$
$$\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$$

In particular, let $t = t_0$, and $t_0 = 0, t_1 = 1$. This becomes:

$$\Phi(0, 0) = \Phi(0, 1)\Phi(1, 0)$$
$$\phi^{-}e^{A} = I$$

and we conclude that $e^{A}$ is invertible for any constant matrix $A$. This is a very important result that we state as a Theorem:

**Theorem 1:** Let $A$ be some $n \times n$ real matrix. Then $e^{A} \triangleq \exp(A)$ is an invertible matrix:

$$\exp(A) : A \rightarrow e^{A} \in \text{Gl}(n, \mathbb{R})$$

Let the set of $n \times n$ skew symmetric matrices (i.e., $A = -A^T$) be denoted by $\text{so}(n)$.

**Proposition:** Let $A \in \text{so}(n)$. Then $e^{A}$ is an orthogonal matrix (i.e., $e^{A} \cdot (e^{A})^T = I$).

**proof:** From Theorem 1 above we know that $e^{A}$ is invertible. In fact,

$$e^{A} \cdot \phi^{-} = I$$

since $-A = A^T$

$$e^{A} \cdot e^{A^T} = e^{A} \cdot (e^{A})^T = I$$

directly from the Peano-Baker series.