ME/SE 740

Lecture 7

An Encounter with Differentiable Manifolds

Last lecture we discussed the concept of homomorphisms. We add to that discussion today by mentioning that for some group $(G, \cdot)$ a homomorphism $h : G \longrightarrow G\ell(n)$ is called a "representation:" Recall that $G\ell(n)$ is the group of $n \times n$ invertible matrices. In particular, $SE(2)$ and $SE(3)$ (the groups of proper rigid motions in the plane and space respectively), have canonical representations:

$$SE(2) : \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} cos\theta & -sin\theta & x \\ sin\theta & cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$SE(3) : \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We now present some basic concepts from the theory of differentiable manifolds. These are relevant in our discussion about motions.

Example: Rigid body motions: $XX^T = I$ where X is the 3×3 matrix (9 parameters, 6 constraints, 3 degrees of freedom):

$$X = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$$

A nice introduction to manifolds is given in: Louis Auslander, Robert MacKenzie, "*Introduction to Differentiable Manifolds*," McGraw-Hill, 1963, Dover reissue.

Definition: A manifold is a pair (M, Φ) where M is a Hausdorf topological space and Φ is a collection of mappings such that:

1. Each $\phi \in \Phi$ maps an open domain dom$\phi \subset M$ to $\mathbb{R}^n$
2. ϕ maps dom$\phi \longrightarrow \mathbb{R}^n$ is $1-1$ and continuous
3. If ϕ, ψ are two elements of Φ and (domϕ) $\cap$ (domψ) is not empty then $\psi \circ \phi^{-1}$ is a differentiable mapping (C^k, k-times differentiable) $\mathbb{R}^n \longrightarrow \mathbb{R}^n$
4. The domains (domϕ) of Φ cover M.

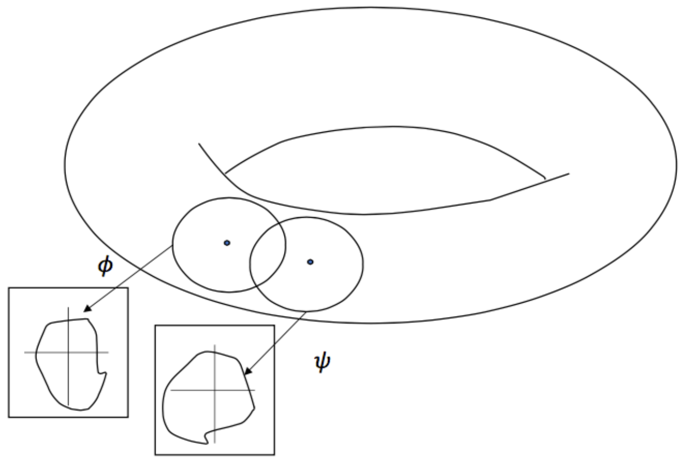

Figure 1: **Manifold**

Locally the spaces look like Euclidean spaces but globally the space is more complex (donut).

A natural question to ask is how do manifolds come up?
Let $p_1, p_2, \cdots p_m$ be functions that map $\mathbb{R}^n$ to $\mathbb{R}^1$. Consider then the set M defined as:

$$M = \{x : p_i(x) = 0, \quad i = 1, 2, \cdots m\}$$

Suppose that the $n \times m$ matrix:

$$\left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial x}, \cdots, \frac{\partial p_m}{\partial x} \right)$$

has rank p, at each point of $M = \{x : p_i(x) = 0, i = 1, 2, \cdots, n\}$. Then M admits the structure of a differentiable manifold of dimension $n - p$.

Example 1: Let $m = 1$, and $p(x_1, x_2, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$. Then the $n \times 1$ matrix:

$$\frac{\partial p}{\partial x} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{pmatrix}$$

on the locus $p(x_1, x_2, \cdots, x_n) = 0$ or $M = \{x : p(x_1, x_2, \cdots, x_n) = 0\}$, has rank 1. The only way the rank of this matrix would be 0 is when $x_i = 0$, for all $i = 1, 2, \cdots, n$, and this point is not an element of M. This defines the $n - 1$ dimensional sphere S^{n-1}.

Important Observation: Let $\bar{p}(x_1, x_2, \cdots, x_n) = (p_1(x_1, x_2, \cdots, x_n), p_2(x_1, x_2, \cdots, x_n), \cdots, p_m(x_1, x_2, \cdots, x_n))$. The condition for $M = \{x : \bar{p}(x_1, x_2, \cdots, x_n) = 0\}$ to be a differential manifold, is that $\frac{\partial \bar{p}}{\partial x}$ has constant rank on M.

Note that if we expand $\bar{p}$ in a power series about any point $x^\star \in M$ then:

$$\bar{p}(x^\star + \epsilon\vec{v}) = \bar{p}(x^\star) + \epsilon\frac{\partial\bar{p}}{\partial x}(x^\star)\vec{v} + h.o.t.$$

The the linear mapping $\vec{v} \longrightarrow \frac{\partial\bar{p}}{\partial x}(x^\star)\vec{v}$ is the derivative of $\bar{p}$ at $x^\star$ whose rank is constant if M is a differentiable manifold.

Example 2: Consider the set of 3×3 matrices X such that $XX^T = I$. Does this define a differentiable manifold?

We show this by using the constructive definition given above (i.e, expressed in the Important Observation given above).

$$(X + \epsilon\delta X)(X + \epsilon\delta X)^T = XX^T + \underbrace{\epsilon X\delta X^T + \epsilon\delta XX^T}_{\epsilon(X\delta X^T + \delta XX^T)} + \epsilon^2\delta X\delta X^T$$

The gradient or derivative terms we are interested in are: $\left(X\delta X^T + \delta XX^T\right)$. So what is the rank of:

$$V \longrightarrow VX^T + XV^T$$

This maps $n\times n$ matrices V to $n\times n$ symmetric matrices $VX^T + XV^T$. Now the rank of this linear mapping is less-than-or-equal to $\frac{n(n+1)}{2}$. In fact, one can show that the rank is equal to to $\frac{n(n+1)}{2}$. Let M be any arbitrary $n\times n$ symmetric matrix and let $V = \frac{MX}{2}$. Then

$$VX^T + XV^T = (\frac{MX}{2})X^T + X(\frac{X^TM}{2}) = \frac{M}{2} + \frac{M}{2} = M$$

This rank is constant and we have shown that the set of $n\times n$ orthogonal matrices is a differentiable manifold of dimension $n^2 - p$. Now, $n^2 - p = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Non-example of a differentiable manifold: Consider the set in the plane $p(x,y) = 0$ where $p(x,y) = y^2 - x^2(x+1)$ (see figure below). We note that the set $M = \{(x,y) : p(x,y) = 0$ does NOT define a differentiable manifold:

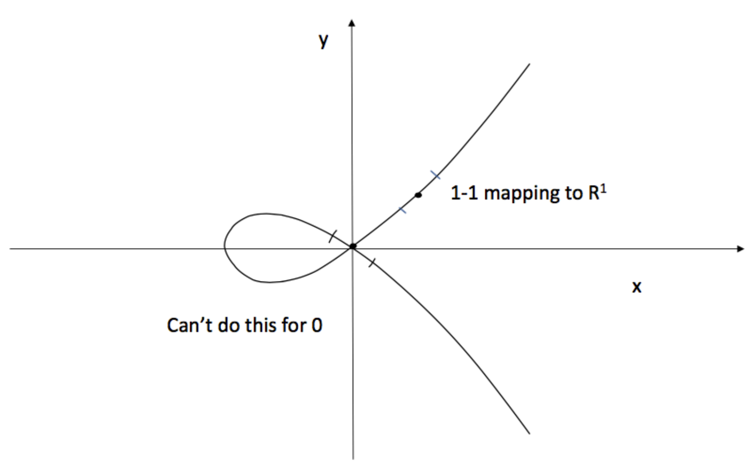

Figure 2: **Not a Differentiable Manifold**

We can compute $\frac{\partial p}{\partial x} = -3x^2 - 2x$, and $\frac{\partial p}{\partial y} = 2y$. This makes the matrix:

$$\begin{pmatrix} \frac{\partial p}{\partial x}(0,0) \\ \frac{\partial p}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which violates the condition of being rank 1 for all points in $p(x,y) = 0$.
Mappings: A mapping $f : M \longrightarrow N$ between differentiable manifolds is C^k-differentiable if $\psi \circ f \circ \phi^{-1}$ is C^k differentiable where $\phi \in \Phi_M$ and $\psi \in \Phi_N$ are the coordinate mappings:

$$\begin{aligned} \phi^{-1} : \quad \mathbb{R}^m &\longrightarrow M \\ f : \quad M &\longrightarrow N \\ \psi : N &\longrightarrow \mathbb{R}^n \end{aligned}$$

Tangent Spaces: Let $p_1, p_2, \cdots p_m : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^1$, and consider $M = \{x : p_i(x) = 0, i = 1, 2, \cdots, m\}$. M is a differentiable manifold of dimension $n - p$ if the $n \times m$ matrix $(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial x}, \cdots, \frac{\partial p_m}{\partial x})$ has rank p at each $x \in M$. The tangent space to M at some point x is:

$$\{\vec{v} \in \mathbb{R}^n : \frac{\partial p_i}{\partial x} \cdot \vec{v} = 0, i = 1, 2, \cdots m\}$$

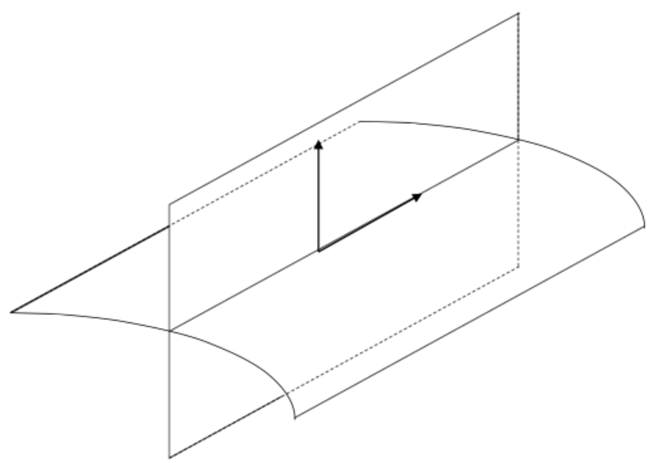

Figure 3: **Tangent Space**

Now the tangent space is the null space (kernel) of the linearization of:

$$\bar{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \quad : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

The null space has dimension $n-p$. Locally tangent spaces are the "same" as manifolds.

$$J \;=\; \begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \ldots & \frac{\partial p_m}{\partial x_1} \\ \frac{\partial p_1}{\partial x_2} & \ldots & \frac{\partial p_m}{\partial x_2} \\ \vdots & & \\ \frac{\partial p_1}{\partial x_n} & \ldots & \frac{\partial p_m}{\partial x_n} \end{pmatrix}$$

The tangent space is: $J^T \cdot \vec{v} \;=\; 0$, (which is the kernel of J^T). If rank of J is p (same as the rank of J^T where $J^T : \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^m$), then the dimension of $kerJ^T \;=\; n-p$.