2-D Rigid Body Motions and Coordinate Transformations

A major component of this course is the study of Kinematic Chains. Consider for example the robotic manipulator shown below:

**Kinematic Chains in 2-D**
In the plane, the configuration of a rigid body is completely described by 3 parameters, $x$, $y$ and $\theta$, giving respectively the position and orientation of frame E with respect to frame B.

In 3-D the configuration of a rigid body is completely described by 6 parameters. Position needs 3 parameters, orientation needs 3 parameters: pitch, roll, yaw.
The motion of taking frame $E_1$ to frame $E_2$ could be parameterized by:

\[
(x(t), y(t), \theta(t)) = (x_1, y_1, \theta_1) + t(x_2 - x_1, y_2 - y_1, \theta_2 - \theta_1)
\]

This works in 2-D but does not “lift” to 3-D.

**Alternative Representation**

The direction of coordinate axis unit vectors in the $E$ frame in terms of $B$ frame coordinates are:

\[
\begin{pmatrix}
\cos \theta \\
-\sin \theta \\
\sin \theta \\
\cos \theta
\end{pmatrix},
\begin{pmatrix}
-\sin \theta \\
\cos \theta
\end{pmatrix}
\]

Putting these together in a matrix generates:

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
The simplest case of representing the E frame in terms of the B frame is when \( \theta = 0 \) (i.e., when E frame coincides with I frame as shown in Figure (5)).

![Image of Figure 5: Point Q Coordinates in Different Frames]

If point \( Q \) has E frame coordinates \( \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E \), then the B frame coordinates satisfy (vector \((x, y)\) is in B-frame coordinates):

\[
\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E + \begin{bmatrix} x \\ y \end{bmatrix}
\]

Since in general \( \theta \neq 0 \), we consider a two-step process to specify \((\tilde{x}, \tilde{y})_B\) in terms of \((\tilde{x}, \tilde{y})_E\):

**Step 1:**

\[
\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_I
\]

where the I frame is depicted in Figure (5) as a translation of the B frame.

**Step 2:**

\[
\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_I = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E
\]

\[
\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E + \begin{bmatrix} x \\ y \end{bmatrix}
\]

The position and orientation of the E frame coordinates in terms of B frame coordinates is given by:

\[
\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}
\]

From the above discussion, this pair specifies B frame coordinates of an arbitrary point \( Q \) in terms of its E frame coordinates.
Another way to look at this pair is that point $Q$ can be obtained from a point $\tilde{Q}$ (see Figure(5)) whose base frame coordinates are numerically equal to $(x, y)_E$, by rotating $\tilde{Q}$ through an angle $\theta$ and then translating it by $(x, y)$.

The result of the rotation is:

$$\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}_E$$

The result of the translation then gives:

$$\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}_B = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}_E + \begin{bmatrix}
x \\
y
\end{bmatrix}$$

Note: Any rigid motion of a plane can be thought of as a rotation followed by a translation.

Let us then use the rigid motion representation point of view.

![Figure 6: Rigid Body Motions](image)

Think of these as rigid motions:

$$T_1 : \begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{bmatrix}, \begin{bmatrix}
x_1 \\
y_1
\end{bmatrix} \quad T_2 : \begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{bmatrix}, \begin{bmatrix}
x_2 \\
y_2
\end{bmatrix}$$

The composite rigid body motion is:

$$T_3 : \begin{bmatrix}
\cos \theta_3 & -\sin \theta_3 \\
\sin \theta_3 & \cos \theta_3
\end{bmatrix}, \begin{bmatrix}
x_3 \\
y_3
\end{bmatrix}$$

Under rigid motion $T_1$, an arbitrary point with coordinate $(x, y)$ moves to:

$$\begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
x_1 \\
y_1
\end{bmatrix}$$

Under rigid motion $T_2$, this arbitrary point moves further to

$$\begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{bmatrix}
\begin{bmatrix}
\left(\begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
x_1 \\
y_1
\end{bmatrix}\right) + \begin{bmatrix}
x_2 \\
y_2
\end{bmatrix}$$
\[
\begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix}
+ \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
+ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} =
\begin{pmatrix}
\cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\
\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2)
\end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix}
+ \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
+ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]

This formula (since the point \((x, y)\) is arbitrary) shows that:

\[
\begin{bmatrix}
\cos \theta_3 & -\sin \theta_3 \\
\sin \theta_3 & \cos \theta_3
\end{bmatrix}
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}
\]

is equivalent to:

\[
\begin{pmatrix}
\cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\
\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2)
\end{pmatrix}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
+ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]

This formula gives a law of composition of rigid motions of the plane: \(T_3 = T_2 \circ T_1\) (rigid motion 1 followed by rigid motion 2).

Rigid motions are invertible. If motions 1, and 2 are inverses then:

\[
\theta_3 = 0, \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

No net rotation, no net translation. In terms of the above formula:

\[
\theta_2 = -\theta_1, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -\begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
= \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix}
\]

Now,

\[
T_2 : \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]

is equivalent to:

\[
\begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & \cos \theta_1
\end{pmatrix}
= \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}^{-1}, \quad -\begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

Important Fact: Rigid motion composition satisfies an associative law \(T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3\).

Dual interpretation of \(T_i\):

\[
T_i : \begin{pmatrix}
\cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i
\end{pmatrix}
\begin{pmatrix} x_i \\ y_i \end{pmatrix}
\]

- Coordinate Transformation
- Rigid Body Motion

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Figure 7: Different Frames

Observation 1: A coordinate frame \( III \) is obtained from a coordinate frame \( II \) by the transformation:

\[
T_2 : \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]

Frame \( II \) has been obtained from frame \( I \) by the transformation:

\[
T_1 : \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

Suppose a point has coordinates \((x, y)\) with respect to frame \( III \), it then has coordinates:

\[
\begin{pmatrix}
\cos \theta_3 & -\sin \theta_3 \\
\sin \theta_3 & \cos \theta_3
\end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}
\]

Note that the frame \( I \) coordinates are also given by:

\[
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix} \begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

That is:

\[
T_3 = T_1 \circ T_2
\]

Observation 2): A point initially at \( Q = (x, y) \) with respect to frame \( I \) moves to a new location under rigid body motion \( T_1 \). It is subsequently moved under new rigid body motion \( T_2 \). The result of composing these rigid body motions is

\[
\bar{T}_3 = T_2 \circ T_1
\]

Note that in general

\[
T_3 \neq \bar{T}_3
\]