ME/SE 740

Lecture 4

2-D Rigid Body Motions and Coordinate Transformations

A major component of this course is the study of Kinematic Chains. Consider for example the robotic manipulator shown below:

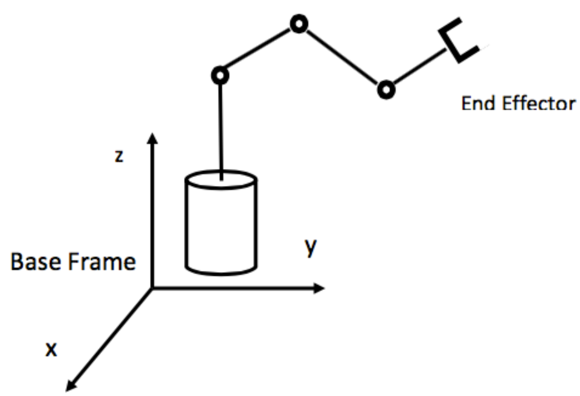

Figure 1: **Kinematic Chain**

Kinematic Chains in 2-D
In the plane, the configuration of a rigid body is completely described by 3 parameters, x, y and θ, giving respectively the position and orientation of frame E with respect to frame B.

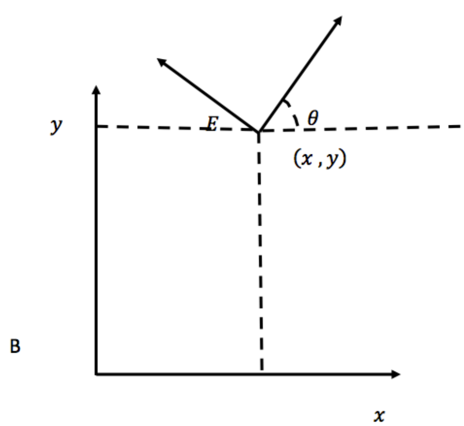

Figure 2: **Base Frame and End Effector Frame**

In 3-D the configuration of a rigid body is completely described by 6 parameters. Position needs 3 parameters, orientation needs 3 parameters: pitch, roll, yaw.

The motion of taking frame E_1 to frame E_2 could be parameterized by:

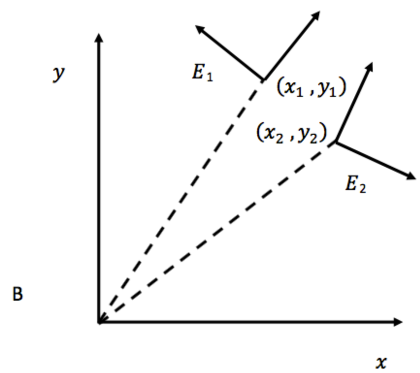

Figure 3: **Motion of Frames**

$$(x(t), y(t), \theta(t)) \quad = \quad (x_1, y_1, \theta_1) + t(x_2 - x_1, y_2 - y_1, \theta_2 - \theta_1)$$

This works in 2-D but does not "lift" to 3-D.

Alternative Representation

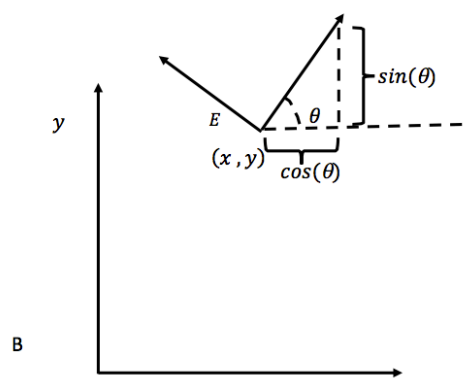

Figure 4: **B and E Frames**

The direction of coordinate axis unit vectors in the E frame in terms of B frame coordinates are:

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Putting these together in a matrix generates:

$$\begin{pmatrix} \cos\theta) & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The simplest case of representing the E frame in terms of the B frame is when $\theta = 0$ (i.e., when E frame coincides with I frame as shown in Figure (5).

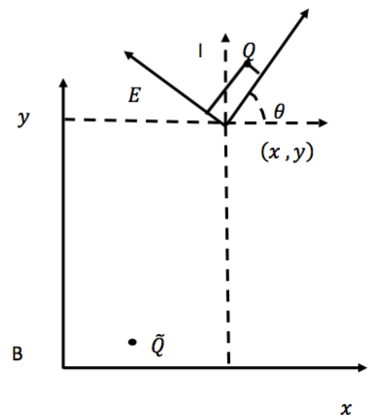

Figure 5: **Point Q Coordinates in Different Frames**

If point Q has E frame coordinates

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E$$

then the B frame coordinates satisfy (vector (x, y) is in B-frame coordinates):

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E + \begin{bmatrix} x \\ y \end{bmatrix}$$

Since in general $\theta \neq 0$, we consider a two-step process to specify $(\tilde{x}, \tilde{y})_B$ in terms of $(\tilde{x}, \tilde{y})_E$:

Step 1:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_I$$

where the I frame is depicted in Figure (5) as a translation of the B frame.

Step 2:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_I = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E + \begin{bmatrix} x \\ y \end{bmatrix}$$

The position and orientation of the E frame coordinates in terms of B frame coordinates is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

From the above discussion, this pair specifies B frame coordinates of an arbitrary point Q in terms of its E frame coordinates.

Another way to look at this pair is that point Q can be obtained from a point $\tilde{Q}$ (see Figure(5)) whose base frame coordinates are numerically equal to $(x, y)_E$, by rotating $\tilde{Q}$ through an angle θ and then translating it by (x, y).

The result of the rotation is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E$$

The result of the translation then gives:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}_E + \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: Any rigid motion of a plane can be thought of as a rotation followed by a translation.

Let us then use the rigid motion representation point of view.

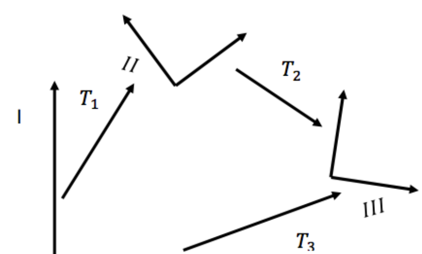

Figure 6: **Rigid Body Motions**

Think of these as rigid motions:

$$T_1 : \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad T_2 : \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

The composite rigid body motion is:

$$T_3 : \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

Under rigid motion T_1, an arbitrary point with coordinate (x, y) moves to:

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Under rigid motion T_2, this arbitrary point moves further to

$$\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \left[\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right] + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1)+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

This formula (since the point (x, y) is arbitrary) shows that :

$$\begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

is equivalent to:

$$\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}, \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

This formula gives a law of composition of rigid motions of the plane: $T_3 = T_2 \circ T_1$ (rigid motion 1 followed by rigid motion 2).

Rigid motions are invertible. If motions 1, and 2 are inverses then:

$$\theta_3 = 0, \qquad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

No net rotation, no net translation. In terms of the above formula:

$$\theta_2 = -\theta_1, \qquad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix}$$

Now,

$$T_2 : \quad \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

is equivalent to:

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}^{-1}, \quad -\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Important Fact: Rigid motion composition satisfies an associative law $T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$.

Dual interpretation of T_i:

$$T_i : \quad \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, \quad \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

- Coordinate Transformation
- Rigid Body Motion

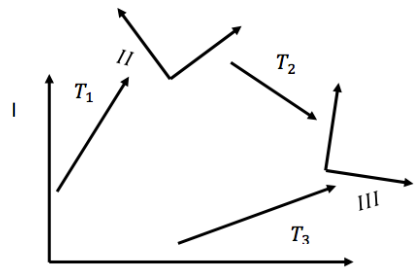

Figure 7: **Different Frames**

Observation 1: A coordinate frame III is obtained from a coordinate frame II by the transformation:

$$T_2 : \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Frame II has been obtained from frame I by the transformation:

$$T_1 : \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Suppose a point has coordinates (x, y) with respect to frame III, it then has coordinates:

$$\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{pmatrix}, \quad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

Note that the frame I coordinates are also given by:

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \left[\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

That is:

$$T_3 = T_1 \circ T_2$$

Observation 2):

A point initially at $Q = (x, y)$ with respect to frame I moves to a new location under rigid body motion T_1. It is subsequently moved under new rigid body motion T_2. The result of composing these rigid body motions is

$$\bar{T}_3 = T_2 \circ T_1$$

.

Note that in general

$$T_3 \neq \bar{T}_3$$