ME/SE 740

Lecture 21

Kinematic Redundancy

Consider once again the 3-link planar manipulator:

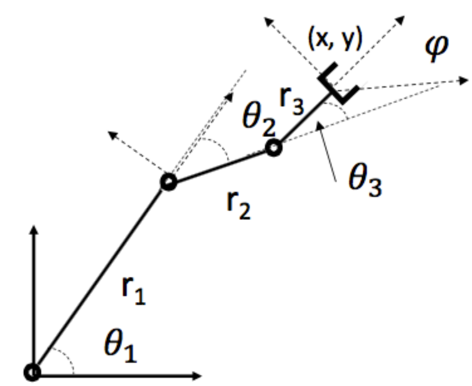

Figure 1: **Three Link Planar Manipulator**

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3) \\ r_1\sin\theta_1 + r_2\sin(\theta_2+\theta_2) + r_3\sin(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

If we are just interested in position of the origin of the tool frame, we have kinematic redundancy:

$$\begin{aligned} J &= \begin{pmatrix} -r_1s_1 - r_2s_{12} - r_3s_{123} & -r_2s_{12} - r_3s_{123} & -r_3s_{123} \\ r_1c_1 + r_2c_{12} + r_3c_{123} & r_2c_{12} + r_3c_{123} & r_3c_{123} \end{pmatrix} \\ &= \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \underbrace{\begin{pmatrix} -r_2s_2 - r_3s_{23} & -r_2s_2 - r_3s_{23} & -r_3s_{23} \\ r_1 + r_2c_2 + r_3c_{23} & r_2c_2 + r_3c_{23} & r_3c_{23} \end{pmatrix}}_{C} \end{aligned}$$

J is non-singular if and only if $\sum_{i=1}^{3}(\det J_i)^2 \; = \; 0$, where j_i is the i^{th} 2×2 minor of C in the above expression.

$$\sum_{i=1}^{3}(\det J_i)^2 \quad = \quad r_1^2(r_2^2 s_2^2 + 2r_2 r_3 s_2 s_{23} + r_3^2 s_{23}^2) \ + \ r_3^2(r_1^2 s_{23}^2 + r_2 s_2 s_{23} + r_2^2 s_3^2) \ + \ r_2^2 r_3^2 s_3^2 \ = 0$$

$$\begin{aligned} &\text{iff} \quad s_3 = 0 \quad \& \quad r_2 s_2 + s_3 s_{23} = 0 \quad \& \quad r_1 s_{23} + r_2 s_3 = 0 \\ &\text{iff} \quad s_2 = 0 \\ &\text{iff} \quad s_2 = s_3 = 0 \end{aligned}$$

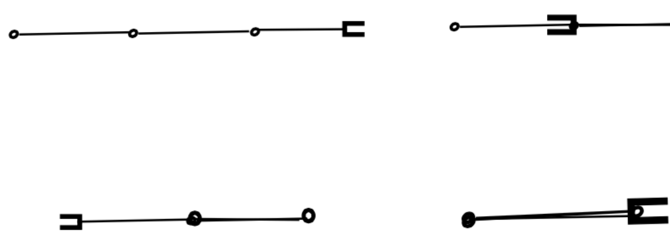

Figure 2: **4 Configurations Corresponding to Singularity**

Resolve velocity control for kinematically redundant mechanisms:

$$x \ = \ f(\theta) \quad \Longrightarrow \quad \dot{x} \ = \ J(\theta)\dot{\theta}$$

How do we solve if J is not square?

Approach 1

$$\dot{\theta} \ = \ J^{\dagger}\dot{x}$$

where $J^{\dagger} \ = \ J^T(JJ^T)^{-1}$, is the Moore-Penrose generalized inverse of J.

Suppose $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation (i.e., an $m \times n$ matrix) and $n > m$. Then there are infinitely many solutions to:

$$\underbrace{A}_{\text{given}}\underbrace{x}_{\text{find}} \quad = \quad \underbrace{y}_{\text{given}}$$

We can restrict the number of solutions by asking for the solution of "minimum norm," i.e., we solve:

$$\min\|x\|^2 \quad \text{subject to} \quad Ax = y$$

Using Lagrange multiplier thinking, find the critical point of:

$$\|x\|^2 \ + \ \lambda^T(Ax - y)$$

The critical point equations in vector form are:

$$\frac{\partial}{\partial x}(\|x\|^2 \ + \ \lambda^T(Ax-y)) \ = \ 2x + A^T\lambda \ = \ 0$$

now multiply the above equation on left by A which leads to:

$$2Ax + AA^T\lambda \ = \ 0, \quad \text{or} \quad 2y + AA^T\lambda \ = 0$$

<u>Claim:</u> If A has rank m then AA^T is invertible.

<u>proof:</u> A has rank m which implies $A^Tx \ = \ 0 \iff x \ = \ 0$. Suppose that $AA^Tx = 0$. Then $x^TAA^Tx \ = \ 0$ and hence $\|A^Tx\|^2 \ = \ 0$. Hence, $A^Tx \ = \ 0$, which implies that $x \ = \ 0$. AA^T is a square matrix whose null space is the zero vector which means that AA^T is invertible.

Hence we can solve $2y + AA^T \ = 0$ for a unique value of λ

$$\lambda \ = \ -2(AA^T)^{-1}y$$

For "free space" motions (no obstacles) the inverse velocity solution is:

$$\dot{\theta} \ = \ J^\dagger \dot{x}$$

with $J^\dagger \ = \ J^T(JJ^T)^{-1}$ yields the minimizing $\dot{\theta}$ (minimizes $\|\dot{\theta}\|^2$) corresponding to $\dot{x}$, (Proposed by Daniel Whitney, 1969).

Problems

1. Klein-Huang (1983) *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 13, showed the Moore-Penrose solution leads to a non-integrable relationship between joint space and tool space configurations. I. e., in general closed curves in tool space do not lead to closed curves in joint space,

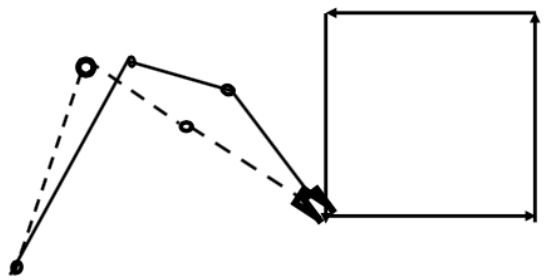

Figure 3: **4 Configurations Corresponding to Singularity**

at the end of the move the joint are not in the same configuration (an undesirable characteristic).

2. Baillieul, Brocket, Hollerback (1984), (1984 CDC), showed that the Moore-Penrose inverse solution did not avoid kinematic singularities.

Demonstration

Let x_0 be any point in the tool space and let $\theta^\star$ be any point in a neighborhood of a singular configuration. Let $x^\star$ be the corresponding tool space point. Choose a workspace trajectory $x(\cdot)$, such that $x(0) \;=\; x^\star$ and $x(1) \;=\; x_0$ and let

$$\dot{\theta} \;=\; J^\dagger \dot{x} \qquad \text{(A)}$$

This will generate the corresponding joint space trajectory with:

$$f(\theta(1)) \;=\; x_0 \quad \text{Call this} \;\; \theta_0 \;=\; \theta(1)$$

Imagine running the trajectory backwards. Consider a tool space trajectory $\tilde{x}(t) \;=\; x(1-t)$. This goes from x_0 to $x^\star$. Consider the joint space trajectory $\tilde{\theta}$ that this corresponds to via:

$$\dot{\tilde{\theta}} \;=\; J^\dagger(\tilde{\theta})\dot{\tilde{x}}, \quad \text{with} \quad \tilde{\theta}(0) \;=\; \theta_0 \qquad (B)$$

Note that for the trajectory $\theta(\cdot)$ defined by (A):

$$\frac{d}{dt}\theta(1-t) \;=\; -J^{\dagger}(\theta(1-t))\dot{x}(1-t) \;=\; J^{\dagger}(\theta(1-t))\dot{\tilde{x}}$$

and by the uniqueness of solutions to ordinary differential equations:

$$\tilde{\theta}(t) \;=\; \theta(1-t), \quad \text{and in particular}$$

$$\tilde{\theta}(t) \;=\; \theta^{\star}$$

the nearly singular configuration we start with.

Hence, since x_0 was arbitrary we have shown that it cannot be assured apriori that the pseudo-inverse technique will generate trajectiries that avoid singularities.

An alternative approach is to find a function $g(\theta(t))$ to maximize, subject to $x(t) = f(\theta(t))$. A necessary condition for maximizing $g(\theta(t))$ subject to $x(t) = f(\theta(t))$ is that:

$$\frac{\partial g(\theta(t))}{\partial \theta} \cdot n_j(\theta(t)) \;=\; 0, \quad \vec{n}_j \in \text{null space of J}$$

If I let $G(\theta) = \frac{\partial g(\theta(t))}{\partial \theta} \cdot n_j(\theta(t))$, then $G(\theta) = 0$ along admissible trajectories. An algorithm to find admissible trajectories is the Extended Jacobian method:

$$\dot{\theta} \;=\; (J_e^{-1}) \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \quad \text{where} \quad J_e \;=\; \begin{pmatrix} J \\ G(\theta) \end{pmatrix}$$

a very successful technique.