ME/SE 740

Lecture 20

Infinitesimal Motions

Recall Euler's Theorem (Lecture 9) where we showed that every 3×3 proper rotation $X \in SO(3)$ is a rotation about an axis $\vec{k}$ by a certain amount θ. We had developed the following expression:

$$R(\theta, \vec{k}) = \begin{pmatrix} k_x^2(1-\cos\theta)+\cos\theta & -k_z\sin\theta + k_xk_y(1-\cos\theta) & k_y\sin\theta + k_xk_z(1-\cos\theta) \\ k_z\sin\theta + k_xk_y(1-\cos\theta) & k_y^2(1-\cos\theta)+\cos\theta & -k_x\sin\theta + k_yk_z(1-cos\theta) \\ -k_y\sin\theta + k_xk_z(1-\cos\theta) & k_x\sin\theta + k_yk_z(1-\cos\theta) & k_z^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$

Replacing $\theta \longrightarrow d\theta$ in the above expression generates:

$$\begin{pmatrix} 1 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{pmatrix} \Longrightarrow dR(\theta, \vec{k}) = \begin{pmatrix} 0 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 0 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 0 \end{pmatrix}$$

Then

$$\begin{aligned} R(x,\delta_x)R(y,\delta_y)R(z,\delta_z) &\sim R(y,\delta_y)R(z,\delta_z)R(x,\delta_x) \sim \text{ in any order} \\ &\sim \begin{pmatrix} 1 & -\delta_z & \delta_y \\ \delta_z & 1 & -\delta_x \\ -\delta_y & \delta_x & 1 \end{pmatrix} \end{aligned}$$

So we can write:

$$dR = \begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix}$$

Putting in translations:

$$\begin{aligned} dT &= \underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta} T \\ &= T \underbrace{\begin{pmatrix} 0 & -\delta_z^T & \delta_y^T & dx^T \\ \delta_z^T & 0 & -\delta_x^T & dy^T \\ -\delta_y^T & \delta_x^T & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta^T} \end{aligned}$$

where Δ, Δ^T are infinitesimal motions and T microscopic, and Δ is in base frame whereas Δ^T is in tool frame.

Suppose that $T = AB$:

$$dT \; = \; \Delta T \; = \; \Delta AB \; = \; A\Delta^A B \; = \; A\Delta^A A^{-1}AB \; = \; A\Delta^A A^{-1}T$$

This implies that: $\Delta \; = \; A\Delta^A A^{-1}$. We can also write: $dT \; = \; T\Delta^T \; = \; AB\Delta^T \; = \; A\Delta^A B \; = \; ABB^{-1}\Delta^A B \; = \; TB^{-1}\Delta^A B$ which implies that $\Delta^T \; = \; B^{-1}\Delta^A B$.

This circle of ideas is key to computing the Jacobian of Kinematics transformations $T \; = \; A_1A_2A_3A_4A_5A_6$.

$$\begin{aligned}
T &= T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \\
\frac{\partial T}{\partial \theta_i} &= A_1A_2\cdots A_{i-1}(\theta_{i-1})\frac{\partial A_i(\theta_i)}{\partial \theta_i}A_{i+1}(\theta_{i+1})\cdots A_6(\theta_6) \\
\frac{\partial A_i(\theta_i)}{\partial \theta_i} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A_i(\theta_i) \\
\frac{\partial T}{\partial \theta_i} &= A_i(\theta_1)\cdots A_{i-1}(\theta_{i-1}) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A_i(\theta_i)\cdots A_6(\theta_6) \\
&= A_1(\theta_1)\cdots A_{i-1}(\theta_{i-1}) \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0_2 \\ \hline \multicolumn{2}{c|}{0_2} & 0_2 \end{array}\right) A_{i-1}(\theta_{i-1})^{-1}\cdots A_1(\theta_1)^{-1}A_1A_2A_3A_4A_5A_6 \\
&= \underbrace{A_1(\theta_1)\cdots A_{i-1}(\theta_{i-1}) \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0_2 \\ \hline \multicolumn{2}{c|}{0_2} & 0_2 \end{array}\right) A_{i-1}(\theta_{i-1})^{-1}\cdots A_1(\theta_1)^{-1}}_{\Delta^i} \; T \\
\frac{\partial T}{\partial \theta_i} &= \Delta^i T
\end{aligned}$$

To compute the Jacobian write:

$$\underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Delta} T \; = \; \sum_{i=1}^{6} \frac{\partial T}{\partial \theta_i}\, d\theta_i \; = \; \sum_{i=1}^{6} \Delta^i T d\theta_i$$

which implies:

$$\Delta \; = \; \sum_{i=1}^{6} \Delta^i d\theta_i$$

Example 4-dof wrist

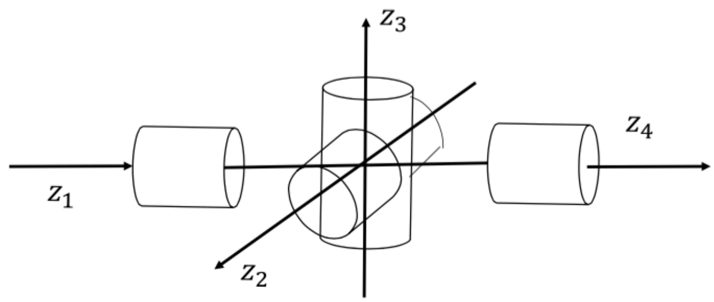

Figure 1: **4-dof wrist**

The DH Parameters are:

i	α_i	a_i	d_i
1	90°	0	0
2	90°	0	0
3	90°	0	0
4	0	0	0

and the A_i and T matrices are:

$$\begin{aligned}
A_1 &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
A_2 &= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
A_3 &= \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
A_4 &= \begin{pmatrix} \cos\theta_4 & -\sin\theta_4 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
T &= A_1 A_2 A_3 A_4 = \underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix}}_{\text{infinitesimals in tool space}}
\end{aligned}$$

Compute the Jacobian (relative to the base frame):

$$
\begin{aligned}
T &= \underbrace{\begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{A}_1}\underbrace{\begin{pmatrix} c_2 & s_2 & 0 \\ 0 & 0 & 0 \\ s_2 & -c_2 & 1 \end{pmatrix}}_{\tilde{A}_2}\underbrace{\begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix}}_{\tilde{A}_3}\underbrace{\begin{pmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{A}_4 = A_4} \\
\frac{\partial T}{\partial \theta_1} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_2} &= \tilde{A}_1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tilde{A}_2\tilde{A}_3\tilde{A}_4 \;=\; \tilde{A}_1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tilde{A}_1^{-1}\underbrace{\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4}_{T} \\
&= \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T \\
&= \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T \;=\; \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ c_1 & s_1 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_3} &= \tilde{A}_1\tilde{A}_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tilde{A}_2^{-1}\tilde{A}_1^{-1} T \\
&= \tilde{A}_1 \begin{pmatrix} 0 & c_2 & 0 \\ -c_2 & 0 & -s_2 \\ 0 & s_2 & 0 \end{pmatrix} \tilde{A}_1^{-1} T \\
&= \begin{pmatrix} s_1c_2 & c_1c_2 & s_1s_2 \\ -c_1c_2 & s_1c_2 & -c_1s_2 \\ 0 & s_2 & 0 \end{pmatrix}\begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T \\
&= \begin{pmatrix} 0 & c_2 & s_1s_2 \\ -c_2 & 0 & -c_1s_2 \\ -s_1s_2 & c_1s_2 & 0 \end{pmatrix} T \\
\frac{\partial T}{\partial \theta_4} &= \tilde{A}_1\tilde{A}_2\tilde{A}_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{A}_3^{-1}\tilde{A}_2^{-1}\tilde{A}_1^{-1} T \\
&= \tilde{A}_1\tilde{A}_2 \begin{pmatrix} 0 & -c_3 & 0 \\ c_3 & 0 & -s_3 \\ 0 & s_3 & 0 \end{pmatrix} \tilde{A}_2^{-1}\tilde{A}_1^{-1} T \\
&= \tilde{A}_1 \begin{pmatrix} 0 & -s_2s_3 & c_3 \\ s_2s_3 & 0 & -s_3c_2 \\ -c_3 & s_3c_2 & 0 \end{pmatrix} \tilde{A}_1^{-1} T \\
&= \underbrace{\begin{pmatrix} 0 & -s_2s_3 & c_1c_3 + s_1c_2s_3 \\ s_2s_3 & 0 & s_1c_3 - c_1c_2s_3 \\ -c_1c_3 - s_1c_2s_3 & -s_1c_3 + c_1c_2c_3 & 0 \end{pmatrix}}_{\text{skew symmetric}} T
\end{aligned}
$$

Now

$$
\begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix} =
$$

$$
\begin{pmatrix} 0 & -d\theta_1 & 0 \\ d\theta_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -c_1 d\theta_2 \\ 0 & 0 & -s_1 d\theta_2 \\ c_1 d\theta_2 & s_1 d\theta_2 & 0 \end{pmatrix} +
$$

$$
\begin{pmatrix} 0 & c_2 d\theta_3 & s_1 s_2 d\theta_3 \\ -c_2 d\theta_3 & 0 & -c_1 s_2 d\theta_3 \\ -s_1 s_2 d\theta_3 & c_1 s_2 d\theta_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & s_2 s_3 & c_1 c_3 + s_1 c_2 s_3 \\ s_2 s_3 & 0 & s_1 c_3 - c_1 c_2 s_3 \\ -c_1 c_3 - s_1 c_2 s_3 & -s_1 c_3 + c_1 c_2 s_3 & 0 \end{pmatrix} d\theta_4
$$

Relate LHS to RHS entries and generate linear equations in the infinitesimals

$$
\begin{pmatrix} \delta_x \\ \delta y \\ \delta z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & s_1 & s_2 c_1 & -s_1 c_3 + c_1 c_2 s_3 \\ 0 & -c_1 & s_2 s_1 & c_1 c_3 + s_1 c_2 s_3 \\ 1 & 0 & c_2 & s_2 s_3 \end{pmatrix}}_{J} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \end{pmatrix}
$$

N. B. does not depend on θ_4

$$
J = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & s_2 & c_2 s_3 \\ 0 & -1 & 0 & c_3 \\ 1 & 0 & -c_2 & s_2 s_3 \end{pmatrix}
$$