ME/SE 740

Lecture 19

Differential Relationships

Let us begin our discussion by considering the planar case and consider the transformation:

$$
\begin{aligned}
T(\theta, x, y) &= \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \\
T(0,0,0) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

We are interested in infinitesimal quantities:

$$
\begin{aligned}
T(d\theta, dx, dy) &= \begin{pmatrix} \cos d\theta & -\sin d\theta & dx \\ \sin d\theta & \cos d\theta & dy \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -d\theta & dx \\ d\theta & 1 & dy \\ 0 & 0 & 1 \end{pmatrix} \;+\; \text{hot}(d\theta, dx, dy)
\end{aligned}
$$

working in the neighborhood of the identity. In the case of infinitesimal motions near any element of $SE(3)$:

$$
\begin{aligned}
dT &= T(\theta + d\theta, x + dx, y + dy) \;-\; T(\theta, x, y) \\
&\quad (\text{use} \cos(a+b) = \cos a \cos b - \sin a \sin b,\;\; \sin(a+b) = \sin a \cos b + \cos a \sin b \;\; \text{and then Taylor Series expansion}) \\
&= \begin{pmatrix} \cos(\theta+d\theta) & -\sin(\theta+d\theta) & x+dx \\ \sin(\theta+d\theta) & \cos(\theta+d\theta) & y+dy \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sin\theta d\theta & -\cos\theta d\theta & dx \\ \cos\theta d\theta & -\sin\theta d\theta & dy \\ 0 & 0 & 0 \end{pmatrix} + \text{hot}(d\theta, dx, dy) - \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\sin\theta d\theta & -\cos\theta d\theta & dx \\ \cos\theta d\theta & -\sin\theta d\theta & dy \\ 0 & 0 & 0 \end{pmatrix} + \text{hot}(d\theta, dx, dy) \\
&= \begin{pmatrix} 0 & -d\theta & dx \\ d\theta & 0 & dy \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} + \text{hot}(d\theta, dx, dy)
\end{aligned}
$$

Now, $T = A_1 A_2 A_3 \cdots A_n$. What do we mean by dT?

1. We could think of a "tiny" motion in the base frame Δ followed by a "macroscopic" (i.e., a transformation that is not tiny) T.

$$\Delta T \;=\; \begin{pmatrix} 1 & -d\theta & dx \\ d\theta & 1 & dy \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$\Delta T \;=\; (\Delta - I)T \;=\; \begin{pmatrix} 0 & -d\theta & dx \\ d\theta & 0 & dy \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

2. We could alternatively make a transformation T followed by an infinitesimal increment Δ_E to get $T \circ \Delta_E$. For this we write:

$$dT \;=\; T(\Delta_E - I) \;=\; \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -d\theta_E & dx_E \\ d\theta_E & 0 & dy_E \\ 0 & 0 & 0 \end{pmatrix}$$

Now consider the 3 dof manipulator shown below:

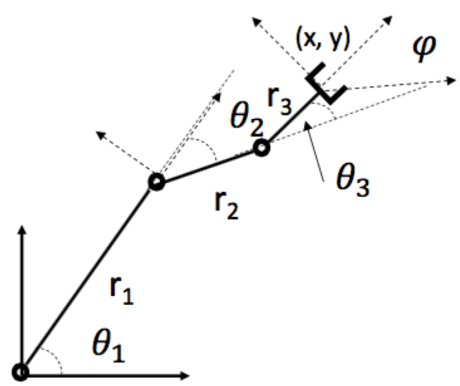

Figure 1: **Planar Case**

$$\begin{pmatrix} x \\ y \\ \phi \end{pmatrix} \;=\; \begin{pmatrix} r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3) \\ r_1\sin\theta_1 + r_2\sin(\theta_2+\theta_2) + r_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{pmatrix}$$

And

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ 1 & 1 & 1 \end{pmatrix}$$

In terms of $SE(2)$ kinematics:

$$T = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & x(\theta_1,\theta_2,\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & y(\theta_1,\theta_2,\theta_3) \\ 0 & 0 & 1 \end{pmatrix}$$

where :

$$\begin{aligned} x(\theta_1,\theta_2,\theta_3) &= r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3) \\ y(\theta_1,\theta_2,\theta_3) &= r_1\sin\theta_1 + r_2\sin(\theta_2+\theta_2) + r_3\sin(\theta_1+\theta_2+\theta_3) \end{aligned}$$

Now

$$\frac{\partial T}{\partial \theta_i} = \begin{pmatrix} -\sin(\theta_1+\theta_2+\theta_3) & -\cos(\theta_1+\theta_2+\theta_3) & \frac{\partial x}{\partial \theta_i} \\ \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & \frac{\partial x}{\partial \theta_i} \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1,2,3$$

In general, if $y = F(x)$ defines a (nonlinear) transformation $F : \mathbb{R}^n \implies \mathbb{R}^n$, then the Jacobian is a linear transformation and has the form:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

A good way to think about this is that $\frac{\partial F}{\partial x} = J$ represents the relationship between infinitesimal quantities dx_i and corresponding infinitesimal quantities dy_i:

$$dy_j = \frac{\partial F_j}{\partial x_1}dx_1 + \frac{\partial F_j}{\partial x_2}dx_2 + \cdots + \frac{\partial F_j}{\partial x_n}dx_n$$

In the case of T above:

$$\begin{aligned} \Delta T &= \begin{pmatrix} 0 & -d\theta & dx \\ d\theta & 0 & dy \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\sin\phi d\phi & -\cos\phi d\phi & dx - y d\phi \\ \cos\phi d\phi & -\sin\phi d\phi & dy + x d\phi \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{\partial T}{\partial \theta_1}d\theta_1 + \frac{\partial T}{\partial \theta_2}d\theta_2 + \frac{\partial T}{\partial \theta_3}d\theta_3 \end{aligned}$$

$$\begin{aligned}
\frac{\partial T}{\partial \theta_i} &= \begin{pmatrix} -s_{123} & -c_{123} & \frac{\partial x}{\partial \theta_i} \\ c_{123} & -s_{123} & \frac{\partial y}{\partial \theta_i} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & \frac{\partial x}{\partial \theta_i} + y \\ 1 & 0 & \frac{\partial y}{\partial \theta_i} - x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Writing:

$$dT = \sum_{i=1}^{3} \frac{\partial T}{\partial \theta_i} d\theta_i, \qquad d\phi = d\theta_1 + d\theta_2 + d\theta_2, \qquad dx = \frac{\partial x}{\partial \theta_1} d\theta_1 + \frac{\partial x}{\partial \theta_2} d\theta_1 + \frac{\partial x}{\partial \theta_3} d\theta_3 + y(d\theta_1 + d\theta_2 + d\theta_3)$$

$$dy = \frac{\partial y}{\partial \theta_1} d\theta_1 + \frac{\partial y}{\partial \theta_2} d\theta_1 + \frac{\partial y}{\partial \theta_3} d\theta_3 - x(d\theta_1 + d\theta_2 + d\theta_3)$$

These equations can be written in matrix form:

$$\begin{pmatrix} dx \\ dy \\ d\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} + y & \frac{\partial x}{\partial \theta_2} + y & \frac{\partial x}{\partial \theta_3} + y \\ \frac{\partial y}{\partial \theta_1} - x & \frac{\partial y}{\partial \theta_2} - x & \frac{\partial y}{\partial \theta_3} - x \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix}$$

Recalling that:

$$\begin{aligned}
x(\theta_1, \theta_2, \theta_3) &= r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2) + r_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y(\theta_1, \theta_2, \theta_3) &= r_1 \sin\theta_1 + r_2 \sin(\theta_2 + \theta_2) + r_3 \sin(\theta_1 + \theta_2 + \theta_3)
\end{aligned}$$

the matrix above gives:

$$\begin{aligned}
\begin{pmatrix} dx \\ dy \\ d\phi \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & r_1 \sin\theta_1 & r_1 \sin\theta_1 + r_2 \sin(\theta_1 + \theta_2) \\ 0 & -r_1 \cos\theta_1 & -r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 & 1 \end{pmatrix}}_{J} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{never singular}} \begin{pmatrix} 0 & 0 & r_2 s_2 \\ 0 & -r_1 & -r_1 - r_2 c_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix}
\end{aligned}$$

Kinematic singularity of the above expression only involves θ_2 (when $\theta_2 = 0$).

We began the lecture with:

$$\begin{pmatrix} 0 & -d\phi_B & dx_B \\ d\phi_B & 0 & dy_B \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -d\phi_E & dx_E \\ d\phi_E & 0 & dy_E \\ 0 & 0 & 0 \end{pmatrix}$$

A formula relating infinitesimals in E frame and B frame.

Lemma: The conjugacy relationship

$$\begin{pmatrix} 0 & -d\phi_T & dx_T \\ d\phi_T & 0 & dy_T \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -d\phi_B & dx_B \\ d\phi_B & 0 & dy_B \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}$$

can be rendered in vector form as:

$$\begin{aligned} \begin{pmatrix} dx_T \\ dy_T \\ d\phi_T \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_B \\ dy_B \\ d\phi_B \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi & -\sin\phi & y \\ \sin\phi & c\phi & -x \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} dx_B \\ dy_B \\ d\phi_B \end{pmatrix} \end{aligned}$$

<u>Spatial Case:</u> Think about infinitesimal rotations about the z, y axes:

$$\underbrace{\begin{pmatrix} 1 & -d\theta & 0 \\ d\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{to first order}} \underbrace{\begin{pmatrix} 1 & 0 & d\phi \\ 0 & 1 & 0 \\ -d\phi & 0 & 1 \end{pmatrix}}_{\text{to first order}} = \underbrace{\begin{pmatrix} 1 & -d\theta & d\phi \\ d\theta & 1 & d\theta d\phi \\ -d\phi & 0 & 1 \end{pmatrix}}_{\text{to first order}}$$

$$\begin{pmatrix} 1 & 0 & d\phi \\ 0 & 1 & 0 \\ -d\phi & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -d\theta & 0 \\ d\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -d\theta & d\phi \\ d\theta & 1 & 0 \\ -d\phi & d\phi d\theta & 1 \end{pmatrix}$$

Where if we "drop" the $d\theta d\phi$ and $d\phi d\theta$ terms, we see that to first order spatial rotations <u>commute</u>. Infinitesimals in $SE(3)$ have the form:

$$\begin{pmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\underbrace{\begin{pmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{pmatrix}}_{\text{infinitesimal rotation matrix}}, \qquad \underbrace{\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{\text{infinitesimal translation}}$$