ME/SE 740

Lecture 18

Elbow Manipulator: Inverse Kinematics Problem Solutions

Recall from Lecture 15 the DH table for the Elbow Manipulator:

i	a_i	α_i	d_i
1	0	90°	0
2	a_2	0	0
3	a_3	0	0
4	a_4	−90°	0
5	0	90°	0
6	0	0	0

Where $T_6^0 = T$ is given by:

$$T = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3)A_4(\theta_4)A_5(\theta_5)A_6(\theta_6) = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with:

$$A_1 = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & 0 & a_i\sin\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad i = 2, 3$$

$$A_4 = \begin{pmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & a_4\cos\theta_4 \\ \sin\theta_4 & 0 & \cos\theta_4 & a_4\sin\theta_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 1: Form the product:

$$A_1^{-1}T \;=\; A_2A_3A_4A_5A_6$$

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \;=\; RHS$$

$$RHS = \begin{pmatrix} c_{234}c_5c_6 - s_{234}s_6 & -c_{234}c_5s_6 - s_{234}c_6 & c_{234}s_5 & a_4c_{234} + a_3c_{23} + a_2c_2 \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & a_4s_{234} + a_3s_{23} + a_2s_2 \\ -s_5c_6 & s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compare the $(3,4)$ entries on both sides:

$$\sin\theta_1 p_x - \cos\theta_1 p_y \;=\; 0 \quad\Longrightarrow\quad \frac{\sin\theta_1}{\cos\theta_1} \;=\; \frac{p_y}{p_x} \;=\; \tan\theta_1$$

There will be two solutions for θ_1 and they differ by π, $(180°)$:

Therefore: $\theta_1 = \text{atan2}(p_x, p_y)$ or $\theta_1 = \text{atan2}(-p_x, -p_y)$

Step 2: Note that no further variables are constrained in writing:

$$A_2^{-1}A_1^{-1}T \;=\; A_3A_4A_5A_6 \quad \text{or} \quad A_3^{-1}A_2^{-1}A_1^{-1}T \;=\; A_4A_5A_6$$

Step 3: Now form:

$$\overline{LHS} \;=\; A_4^{-1}A_3^{-1}A_2^{-1}A_1^{-1}T \;=\; A_5A_6 \;=\; \overline{RHS}$$

Where,

$$\overline{RHS} \;=\; \begin{pmatrix} c_5c_6 & -c_5s_6 & s_5 & 0 \\ s_5c_6 & -s_5s_6 & -c_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\overline{LHS}$ has the form (some terms are not explicitly written):

$$\overline{LHS} = \begin{pmatrix} \star & \star & a_xc_1c_{234} + a_ys_1c_{234} + a_zs_{234} & p_xc_1c_{234} + p_ys_1c_{234} + p_zs_{234} - a_2c_{23} - a_3c_4 - a_4 \\ \star & \star & -a_xs_1 + a_yc_1 & -p_xs_1 + p_yc_1 \\ -n_xc_1s_{234} - n_ys_1s_{234} + n_zc_{234} & -o_xc_1s_{234} - o_ys_1s_{234} + o_zc_{234} & -a_xc_1s_{234} - a_ys_1s_{234} + a_zc_{234} & \star \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparing the $(3,3)$ entries on both sides we see that:

$$\begin{aligned}
a_x c_1 s_{234} + a_y s_1 s_{234} - a_z c_{234} &= 0 \\
(a_x c_1 + a_y s_1) s_{234} &= a_z c_{234} \\
\Longrightarrow \theta_2 + \theta_3 + \theta_4 &= \operatorname{atan2}(a_x c_1 + a_y s_1, a_z)
\end{aligned}$$

Step 4: We now return to Step 1 (where we compared entries in $A_1^{-1}T = A_2A_3A_4A_5A_6$ and examine entries $(1,4)$ and $(2,4)$:

$$\begin{aligned}
a_4 c_{234} + a_3 c_{23} + a_2 c_2 &= p_x c_1 + p_y s_1 \\
a_4 s_{234} + a_3 s_{23} + a_2 s_2 &= p_z
\end{aligned}$$

This is just like the planar manipulator problem we solved earlier which allows us to write:

$$\begin{aligned}
a_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3) &= \bar{x} \\
a_2 \sin\theta_2 + a_3 \sin(\theta_2 + \theta_3) &= \bar{y}
\end{aligned}$$

$$\text{where} \quad \bar{x} = p_x c_1 + p_y s_1 - a_4 c_{234} \qquad \text{and} \qquad \bar{y} = p_z - a_4 s_{234}$$

Using the "cosine law" we can write:

$$\begin{aligned}
\cos\theta_3 &= \frac{\bar{x}^2 + \bar{y}^2 - a_2^2 - a_3^2}{2a_2a_3} \\
\sin\theta_3 &= \pm\sqrt{1 - \cos^2\theta_3}
\end{aligned}$$

For each of these solutions (8 of them) write:

$$\begin{pmatrix} a_2 + a_3\cos\theta_3 & -a_3\sin\theta_3 \\ a_3\sin\theta_3 & a_2 + a_3\cos\theta_3 \end{pmatrix} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

now let $\theta_2 = \tan2(\zeta_1, \zeta_2)$ where:

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \frac{1}{a_2^2 + a_3^2 + 2a_2a_3\cos\theta_3} \begin{pmatrix} a_2 + a_3\cos\theta_3 & a_3\sin\theta_3 \\ -a_3\sin\theta_3 & a_2 + a_3\cos\theta_3 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Step 5:

θ_5 is found by equating entries $(1,3)$ and $(2,3)$ of the expression in Step 3, $\overline{LHS} = \overline{RHS}$, which results in:

$$\begin{aligned}
\cos\theta_5 &= a_x s_1 - a_y c_1 \\
\sin\theta_5 &= a_x c_1 c_{234} + a_y s_1 c_{234} + a_z s_{234}
\end{aligned}$$

Step 6: θ_6 is obtained by comparing $(3,1)$, $(3,2)$ of the expression in Step 3, $\overline{LHS} = \overline{RHS}$, which results in:

$$\begin{aligned}\cos\theta_6 &= -\,o_x c_1 s_{234} - o_y s_1 s_{234} + o_z c_{234}\\ \sin\theta_6 &= -\,n_x c_1 s_{234} - n_y s_1 s_{234} + n_z c_{234}\end{aligned}$$

Closed form solutions can also be found for the PUMA 560 (see CRAIG in the references).

<u>Differential Relationships:</u>

Let

$$\begin{aligned}x &= f(\theta) \quad (e.g.,\ \ f: T^6\ = S^1\times S^1\times S^1\times S^1\times S^1\times S^1 \ \ \longrightarrow\ \ SE(3),\ \ S^1 \text{ is the unit circle})\\ \dot{x} &= \frac{\partial f(\theta)}{\partial\theta}\dot{\theta}\\ \dot{\theta} &= \left(\frac{\partial f(\theta)}{\partial\theta}\right)^{-1}\dot{x}\end{aligned}$$

Note that in the above the inverse computation is “doable” in most cases.

The only problem with this would occur if $\frac{\partial f}{\partial\theta}$ is singular (recall the singular configurations of the planar 2-link manipulator).

$$\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} f_1(\theta_1,\theta_2)\\ f_2(\theta_1,\theta_2)\end{pmatrix} = \begin{pmatrix} r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2)\\ r_1\sin\theta_1 + r_2\sin(\theta_1+\theta_2)\end{pmatrix}$$

Then:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial\theta_1} & \frac{\partial f_1}{\partial\theta_2}\\ \frac{\partial f_2}{\partial\theta_1} & \frac{\partial f_2}{\partial\theta_2}\end{pmatrix} = \begin{pmatrix} -r_1\sin\theta_1 - r_2\sin(\theta_1+\theta_2) & -r_2\sin(\theta_1+\theta_2)\\ r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) & r_2\cos(\theta_1+\theta_2)\end{pmatrix}$$

That makes:

$$\begin{aligned}detJ &= r_1 r_2(\sin(\theta_1+\theta_2)\cos\theta_1 \ - \ \sin\theta_1\cos(\theta_1+\theta_2))\\ &= r_1 r_2\sin\theta_2\end{aligned}$$

Kinematic singularity exists when $\theta_2 \ = \ 0$, or $\theta_2 \ = \ 180°$.

<u>Note</u>: Kinematic singularities are configurations where the Jacobian function looses rank:

Problem: Consider the 2-link manipulator in the figure below at the (near singularity) configurations shown. We assume the two links have the same length $r_1 \ = \ r_2$. We wish to move the tip at unit velocity from position $(\epsilon, 0)$ (which corresponds to the manipulator shown in the “solid” configuration) to position $(0, \epsilon)$, (which corresponds to the manipulator shown in the “dashed” configuration) a distance of $\sqrt{2}\epsilon$. For this to happen $\dot{\theta}$ would need to change <u>very rapidly</u>.

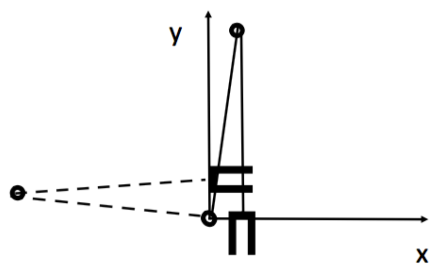

Figure 1: **Two Configurations**

Under diffeomorphic changes of coordinates the form of the Jacobian changes but singularities are invariant. Example, given a nonlinear function $y = f(x)$, consider mutually inverse diffeomorphisms $u = \phi(x), \quad x = \psi(u)$. Let $g(u) = f \circ \psi(u)$:

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}(\psi(u))\frac{\partial \psi}{\partial u}$$

This is singular if and only if $\frac{\partial f}{\partial x}$ is singular since $\frac{\partial \psi}{\partial u}$ is never singular.