ME/SE 740

Lecture 16

Homogeneous Transformations, $SE(3)$ Elements and D-H Parameters

Consider the coordinate frames associated with two consecutive links of some kinematic chain shown below. The k^{th} coordinate frame is determined axes x_k, y_k, z_k with its associated DH-Parameters:

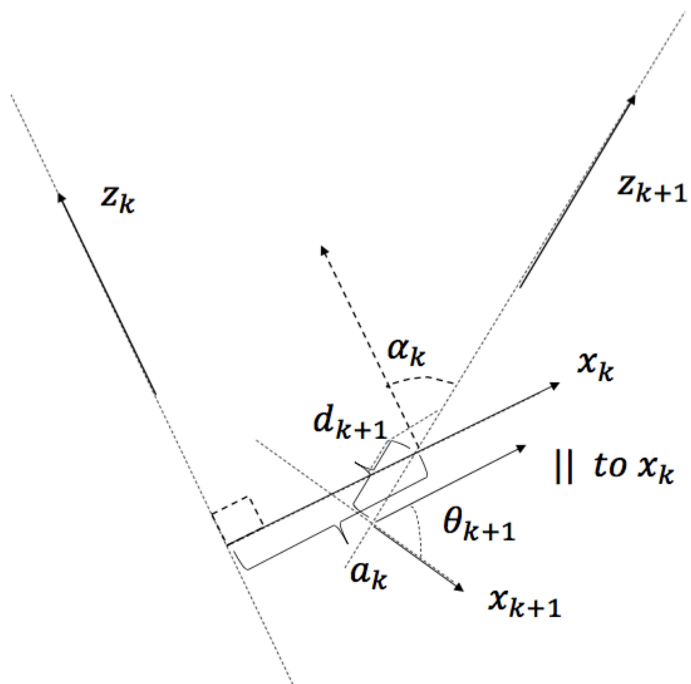

Figure 1: **Link frames from two consecutive links**

Question: How is a point referred to in the k^{th} frame described in the $k+1$ frame?

We answer this question by defining intermediate coordinate systems R, Q, P, S as follows:

- <u>Frame R:</u> It differs from frame k by a translation of a_k units along axis x_k
- <u>Frame Q:</u> It differs from frame R by a rotation α_k about axis x_k
- <u>Frame P:</u> It differs from frame Q by a translation d_{k+1} along axis z_{k+1}
- <u>Frame S:</u> It differs from frame P by a rotation θ_{k+1} about axis z_{k+1}

The matrices that represent the coordinate transformations are:

$$
\begin{aligned}
R &= \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_k & -\sin\alpha_k & 0 \\ 0 & \sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
P &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{k+1} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \cos\theta_{k+1} & -\sin\theta_{k+1} & 0 & 0 \\ \sin\theta_{k+1} & \cos\theta_{k+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Given a point with coordinates $x_{k+1}, y_{k+1}, z_{k+1}$ in the $k+1$ frame, the coordinates in the k^{th} frame are given by the formula:

$$
\begin{pmatrix} x_k \\ y_k \\ z_k \\ 1 \end{pmatrix} = RQPS \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ 1 \end{pmatrix}
$$

<u>Note:</u>:

$$
\begin{aligned}
RQ &= \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & \cos\alpha_k & -\sin\alpha_k & 0 \\ 0 & \sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Screw}(X_k, a_k, \alpha_k) \\
PS &= \begin{pmatrix} \cos\theta_{k+1} & -\sin\theta_{k+1} & 0 & 0 \\ \sin\theta_{k+1} & \cos\theta_{k+1} & 0 & 0 \\ 0 & 0 & 1 & d_{k+1} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1}) \\
\vec{P}_k &= \text{Screw}(X_k, a_k, \alpha_k)\text{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1})\vec{P}_{k+1}
\end{aligned}
$$

Once the joint frames have been defined and the corresponding parameters found the joint frame transformation can be multiplied together to find a single transformation that relates frame $\{N\}$ (at the tip frame) to frame $\{0\}$ (at the base frame):

$$
T_N^0 = T_1^0 T_2^1 \cdots T_N^{N-1}
$$

Recall from our convention that $\alpha_0 = 0, a_0 = 0$ hence we may also write that:

$$T_N^0 = A_1 A_2 \cdots A_N$$

where:

$$\begin{aligned}
A_k &= \mathrm{Screw}(Z_k, d_k, \theta_k)\mathrm{Screw}(X_k, a_k, \alpha_k) \\
&= \begin{pmatrix} \cos\theta_k & -\sin\theta_k & 0 & 0 \\ \sin\theta_k & \cos\theta_k & 0 & 0 \\ 0 & 0 & 1 & d_k \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & \cos\alpha_k & -\sin\alpha_k & 0 \\ 0 & \sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_k & -\sin\theta_k\cos\alpha_k & \sin\theta_k\sin\alpha_k & a_k\cos\theta_k \\ \sin\theta_k & \cos\theta_k\cos\alpha_k & -\cos\theta_k\sin\alpha_k & a_k\sin\theta_k \\ 0 & \sin\theta_k & \cos\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$