ME/SE 740

Lecture 14

Types of Lie Subgroups of $SE(3)$

Theorem (Loncaric) A constraint on mechnism kinematics, $f(g) = 0$, is left (or right) invariant if and only if $f^{-1}(0)$ is a subgroup of $SE(3)$. Hence, the Lie subgroup types of $SE(3)$ are of great interest.

Definition A subgroup J of $SE(3)$ will be called a joint subgroup if there is a neighborhood U of the identity in $SE(3)$ and a pair of rigid bodies in contact such that inside U the set of all possible relative motions is identical to J.

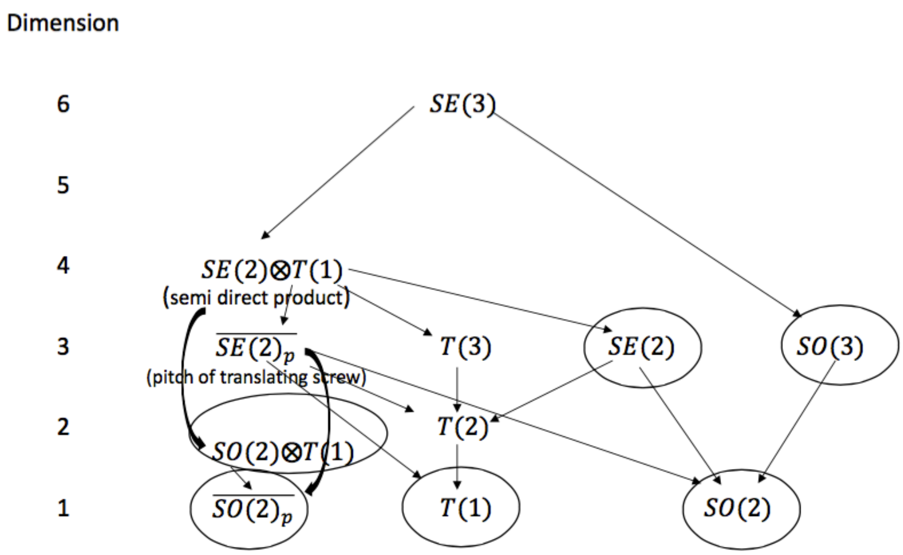

Figure 1: **Lie Subgroup Types**

Theorem: The only types of joint subgroups are $T(1), SO(2), \overline{SO(2)}_p$ (the covering group of screw motions), $SO(2) \bigotimes T(1), S\overline{E(2)}$ and $SO(3)$.

proof: The following arguments restrict the possibilities. First notice that if a body B can be translated (at least locally) in all three directions then the space swept-out must exclude the constraining body and yet be large enough to allow rotations about any axis. Therefore, if a joint subgroup included $T(3)$, it must be all of $SE(3)$, and this is ruled out if we are talking about constrained motions. Similarly if B can be translated freely in any plane, then the free space swept out by B must allow rotation about some perpendicular axis. Therefore, a joint subgroup containing $T(2)$ must include $SE(2)$ as well.

These two observations exclude $T(2), T(3), \overline{SE(2)}_p$, and $SE(2) \bigotimes T(1)$ from cosideration. The remaining subgroups can be realized as joint subgroups.

Note: Lower pairs are exactly types of joint subgroups.

Product of Exponentials

Consider a single strand kinematic chain:

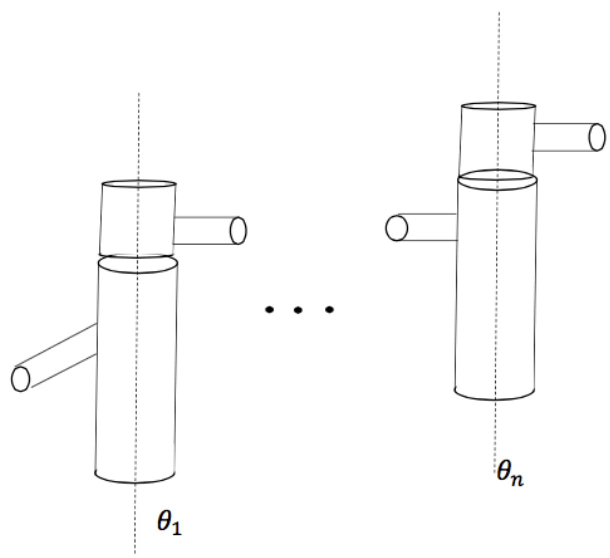

Figure 2: **Kinematic Chain**

If we affix a right-handed triad of orthogonal vectors to the hinge point of each link, the the element of the group that describes the position and orientation of the $i+1^{st}$ link in terms of the i^{th} is:

$$\begin{pmatrix} A_i & b_i \\ 0 & 1 \end{pmatrix} e^{\left(\begin{pmatrix} S_i & 0 \\ 0 & 0 \end{pmatrix} \theta_i \right)}$$

where the rotation is the allowed motion.

In terms of this labeling, the position and orientation of the triad at the free end of the chain is related to the coordinate system at the base by:

$$T(\theta_1, \cdots \theta_n) = M_1 e^{\left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \theta_1 \right]} M_2 e^{\left[\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix} \theta_2 \right]} \cdots M_n e^{\left[\begin{pmatrix} S_n & 0 \\ 0 & 0 \end{pmatrix} \theta_n \right]}$$

where

$$M_i = \begin{pmatrix} A_i & b_i \\ 0 & 1 \end{pmatrix}$$

θ_i are the motion parameters and A_i, b_i are the structural parameters. Since $Pe^RP^{-1} = e^{PRP-1}$ we can use the identity $Me^R = e^{MRM-1}M$ to write:

$$
\begin{aligned}
T(\theta_1, \cdots \theta_n) &= Me^{H_1\theta_1}e^{H_2\theta_2}\ldots e^{H_n\theta_n} \\
&= M_1 e^{\left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}\theta_1\right]} M_2 e^{\left[\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}\theta_2\right]} \cdots M_n e^{\left[\begin{pmatrix} S_n & 0 \\ 0 & 0 \end{pmatrix}\theta_n\right]} \\
&= M_1 M_2 e^{\left[M_2^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}M_2\theta_1\right]} e^{\left[\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}\theta_2\right]} M_3 \cdots M_n e^{\left[\begin{pmatrix} S_n & 0 \\ 0 & 0 \end{pmatrix}\theta_n\right]} \\
&= M_1 M_2 e^{\left[M_2^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}M_2\theta_1\right]} M_3 e^{\left[M_3^{-1}\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}M_3\theta_2\right]} \cdots M_n e^{\left[\begin{pmatrix} S_n & 0 \\ 0 & 0 \end{pmatrix}\theta_n\right]} \\
&= etc.
\end{aligned}
$$

Questions for today:

1. How do we assign systematically coordinate frames to each type of link and joint?
2. Does the method of assignment differ from one type of joint to another?
3. Can we find key design parameters emerging from the group theory we have discussed?

Consider the two reference frames from consecutive links in some kinematic chain:

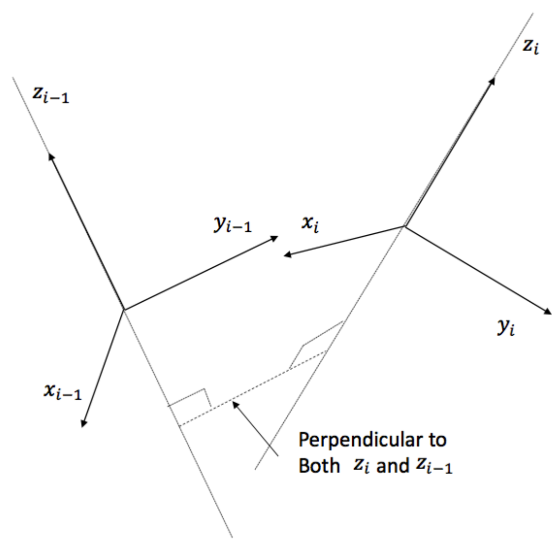

Figure 3: **Consecutive Coordinate Frames**

Specifying z_i in terms of z_{i-1} requires 4 parameters. Specify x_i by choosing:

1. normal direction to z_i (1 dof)
2. a z coordinate along z_i where x_i is attached (1 dof)

A link is a rigid body that defines a relationship between two neighboring joint axes. Given a coordinate frame associated with axis i there are 4 degrees of freedom in specifying axis $i+1$. How do we define joint axes for each lower paired joint?

1.	Revolute $(SO(2))$	z_i is the axis of revolution
2.	Prismatic $(T(1))$	z_i is the axis of translation
3	Screw $(\overline{SO(2)}_p)$	z_i is the axis of motion
4.	Cylindrical $(SO(2) \bigotimes T(1))$	z_i is the axis of motion
5.	Planar $(SE(2))$	z_i is normal to the motion and arbitrary placement in the plane
6.	Spherical $(SO(3))$	z_i is arbitrary passing through the center of rotation

Assigning coordinate frames to joint of Lower Pairs.

Given lower pair joints, assign axes z_i consistently with above table. For any two axes in 3-space there is well defined perpendicular distance between them ($\perp$ segment between axes is not unique if they are parallel but distance is well defined). The amount of distance between z_{i-1} and z_i is a_i and is called the link length. The axis x_{i-1} is defined by the unique direction from z_{i-1} to z_i if these are skew. When $a_{i-1} = 0, x_i = z_{i-1} \bigotimes z_i$. When z_{i-1} is parallel to z_i there is some arbitrariness in choosing the x_{i-1} axis (more about this in the next lecture). In addition to specifying how far z_i is from z_{i-1}, we must say how much it is twisted about the x_{i-1} axis, called α_i (see below):

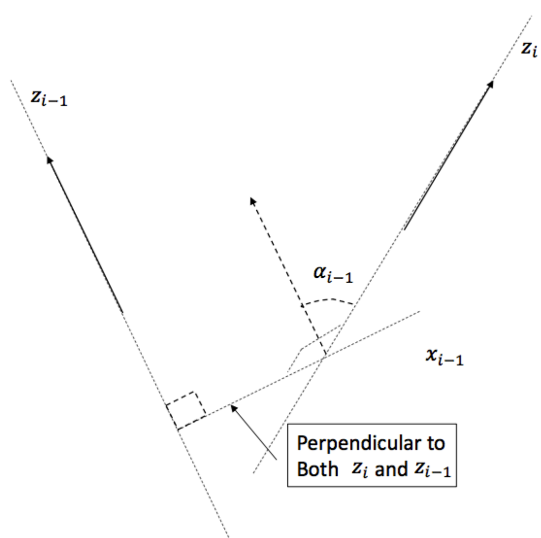

Figure 4: **Defining** α_{i-1}