ME/SE 740

Lecture 13

Chasle's Theorem

Rigid body motion in the plane. Consider a rigid body motion from frame B_1 to frame E. If this motion is not a pure translation, there is a point on the bisector (see figure below) equidistant from the origins of B_1, E the origin of frame B such that the motion of the body about this frame B (parallel to frame B_1) is a pure rotation about that point.

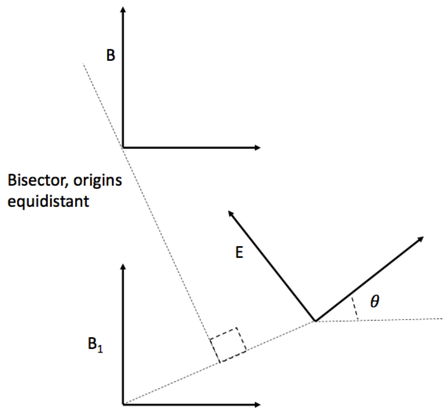

Figure 1: **Rigid body motion in the plane**

In the figure below we show this operation for a triangle QPC where $\theta = 90°$ and a translation $[0, 8]^T$.

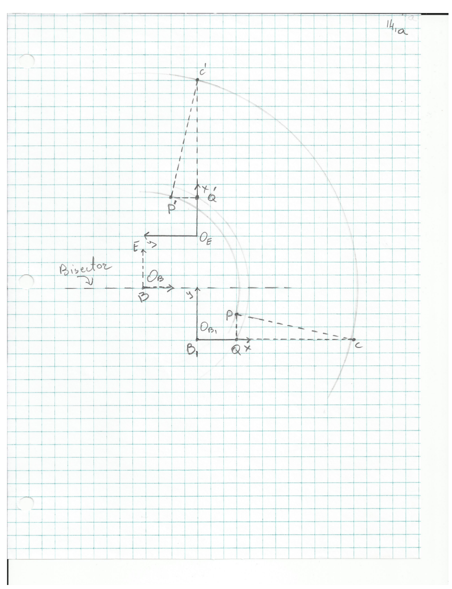

Figure 2: **A Specific Example**

Suppose the motion $B_1 \quad \longrightarrow \quad E$ with respect to the B_1 frame is:

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

An arbitrary point Q moves such that:

$$\underbrace{Q}_{w.r.t. \quad B_1} = \underbrace{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} Q}_{w.r.t. \quad B_1}$$

If we write this expression with respect to the B frame we would write:

$$\underbrace{\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix} Q}_{\bar{Q}} \implies \begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} Q$$

This implies that:

$$\bar{Q} \implies \underbrace{\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}} \bar{Q}$$

The motion $B_1 \quad \longrightarrow \quad E$ with respect to B_1 coordinates is given by:

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

The origin of the B is on the bisector (see figure 1) and is found by working with one point on the rigid body (say vertex Q of the triangle in Fig. 2) in a way such that Q' is obtained by a simple rotation. The other points can be seen to rotate the same way.

Let us work with point Q:

$$\underbrace{Q}_{w.r.t. \quad B_1} = \underbrace{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} Q}_{Q' \quad w.r.t. \quad B_1}$$

Express Q and Q' in B-frame coordinates:

$$\underbrace{\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}}_{\bar{Q}}, \quad \begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} Q$$

(Ψ can be taken to be the I). Now,

$$Q = \begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}^{-1} \bar{Q}$$

In B-frame coordinates Q and Q':

$$\bar{Q}, \quad \underbrace{\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}} \bar{Q}$$

are related by a rotation. In particular:

$$\bar{Q} \longrightarrow \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \bar{Q}$$

Since,

$$\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Psi^T & -\Psi^T d \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi^T & -\Psi^T d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$$

this implies:

$$\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Psi A \Psi^T & -\Psi A \Psi^T d + \Psi b + d \\ 0 & 1 \end{pmatrix}$$

However, the orientation of the B frame with respect to the B_1 frame is arbitrary. Hence take $\Psi = I$. Then, the equation is $(I - A)d + b = 0$, or $b = -(I - A)^{-1} d$. This yields a unique d if and only if $(I - A)^{-1}$ exists, if and only if, the rotation matrix $A \neq I$.

Theorem: (Chasle's Theorem)

For every 4×4 matrix of the form:

$$M = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, \quad A \in SO(3)$$

1. There exists a 4×4 matrix N (proved last lecture) of the form:

$$\begin{pmatrix} S & x \\ 0 & 0 \end{pmatrix}, \quad S = -S^T, \text{such that } \; M = e^{Nt}|_{t-1}$$

2. There exists a 3×3 matrix S, $\; S = -S^T$ such that for:

$$R(t) = e^{\left[\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} t\right]} \begin{pmatrix} I & A^{-1} b \\ 0 & 1 \end{pmatrix}, \quad M = R(1)$$

3. There exists $\Psi \in SO(3)$ and vector d such that

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Psi & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi^T & -\Psi^T b \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

Every motion can be thought of as a translation followed by a rotation about a line passing through a preassigned fixed point: Then the motion is: $\underbrace{R}_{rotation} \circ \underbrace{T}_{translation}$.
Screw motions are special cases that from the proper frame of reference are written as:

$$\begin{aligned}
\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Clearly, these two matrices commute! We may also write:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{\left[\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}\theta\right]}$$

Mechanisms

There are 6 "Lower Pair" joints. Lower Pair joints connect two rigid bodies that share two-dimensional surfaces (see handout). These are (dof is shorthand for degrees of freedom):

1.	Revolute	1 dof motion
2.	Prismatic	1 dof motion
3	Screw	1 dof motion
4.	Cylindrical	2 dof motion
5.	Spherical	3 dof motion
6.	Planar	3 dof motion

The group $SE(3)$ has a number of types of Lie subgroups:

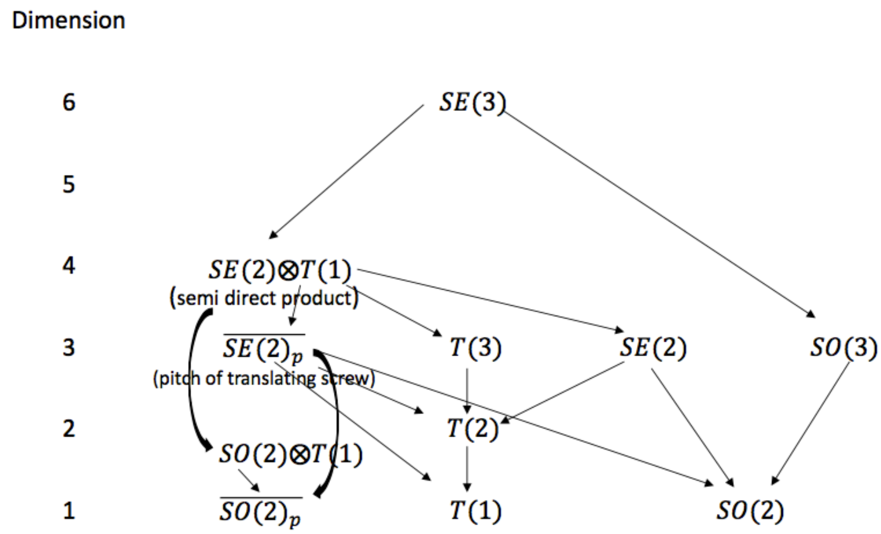

Figure 3: **Lie Subgroup Types**

$T(1)$	translations in some direction	dim 1
$SO(2)$	rotations about some axis	dim 1
$\overline{SO(2)}_p$	screw motions about some axis	dim 1
$SO(2) \bigotimes T(1)$	group of rotations about some axis combined with translations along some axis	dim 2
$SO(3)$	proper spatial rotation	dim 3
$T(2)$	translations in some plane	dim 2
$T(3)$	translations is space	dim 3
$SE(2)$	rigid motion in some plane	dim 3
$\overline{(SE(2)}_p$	translations parallel to some plane together with screw motion perpendicular to the plane	dim 3
$SE(2) \bigotimes T(1)$	motions in the plane together with translations perpendicular to the plane	dim 4

Definition A subgroup J of $SE(3)$ will be called a joint subgroup if there is a neighborhood U of the identity in $SE(3)$ and a pair of rigid bodies in contact such that inside U the set of all possible relative motions is identical to J.