ME/SE 740

Lecture 12

Properties of Exponential Maps

Review of last lecture

There is a $1-1$ correspondence between $so(3)$ and $\mathbb{R}^3$:

$$\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Establish a formal isomorphism:

$$\wedge \ \mathbb{R}^3 \longrightarrow so(3) \quad \text{and} \quad \vee : \ so(3) \longrightarrow \mathbb{R}^3$$

$$\hat{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^{\wedge} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Let $v \in \mathbb{R}^3$, $w \in \mathbb{R}^3$, Ω a 3×3 skew symmetric matrix and define:

$$\begin{pmatrix} v \\ w \end{pmatrix}^{\wedge} = \begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} v \\ \Omega^{\vee} \end{pmatrix}$$

Proposition: Given

$$\xi = \begin{pmatrix} v \\ w \end{pmatrix}, \quad \hat{\xi} = \underbrace{\begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}}_{\in \ so(3)}$$

and $\theta \in \mathbb{R}^3$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$ (i.e., the exponential map takes $se(3)$ into $SE(3)$).

proof 1: (brute force using the Peano-Baker series).

$$
\begin{aligned}
e^{\hat{\xi}\theta} &= I + \begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}\theta + \frac{1}{2}\begin{pmatrix} \hat{w}^2 & \hat{w}v \\ 0 & 0 \end{pmatrix}\theta^2 + \frac{1}{3!}\begin{pmatrix} \hat{w}^3 & \hat{w}^2 v \\ 0 & 0 \end{pmatrix}\theta^3 + \cdots \\
&= \begin{pmatrix} I + \hat{w}\theta + \frac{1}{2}\hat{w}^2\theta^2 + \cdots & v\theta + \frac{1}{2}\hat{w}v\theta^2 + \frac{1}{3!}\hat{w}^2 v\theta^3 + \ldots \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{\hat{w}\theta} & v\theta + \cdots \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Hence, $e^{\hat{\xi}\theta} \in SO(3)$ is of the form we seek proving the Proposition.

proof 2: (more complicated but more insightful).

spacial case i) $w = 0$. Then:

$$
\hat{\xi} = \begin{pmatrix} \hat{0} & v \\ 0 & 0 \end{pmatrix}, \qquad \hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0
$$

$$
e^{\hat{\xi}\theta} = I + \hat{\xi}\theta = \begin{pmatrix} I & v\theta \\ 0 & 1 \end{pmatrix}
$$

Therefore, $e^{\hat{\xi}\theta} \in SE(3)$

spacial case ii) $w \neq 0, \quad \|w\| = 1$. This is not so special as it may always be obtained by appropriate scaling of θ. Let

$$
g = \begin{pmatrix} I & w \times v \\ 0 & 1 \end{pmatrix}
$$

Then:

$$
g^{-1}\hat{\xi}g = \begin{pmatrix} I & -w \times v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & w \times v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{w} & w \times (w \times v) + v \\ 0 & 0 \end{pmatrix}
$$

where we used that $w \times v = \hat{w}v$. In addition, note that:

$$
\begin{aligned}
\hat{w}[w \times (w \times v) + v] &= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}\left[\begin{pmatrix} -w_2^2 - w_3^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & -w_1^2 - w_3^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & -w_1^2 - w_2^2 \end{pmatrix}\vec{v} + \vec{v}\right] \\
&= \hat{w}[-\vec{v} + \vec{v}] = 0
\end{aligned}
$$

where we used the fact that $\hat{w}^3 = -\|w\|^2\hat{w}$ and $\|\hat{w}\|^2 = 1$. Hence:

$$\begin{aligned} e^{g^{-1}\hat{\xi}g} &= I + \begin{pmatrix} \hat{w} & w \times (w \times v) + v \\ 0 & 0 \end{pmatrix}\theta + \frac{1}{2}\begin{pmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{pmatrix}\theta^2 + \cdots \\ &= \begin{pmatrix} e^{\hat{w}\theta} & [w \times (w \times v) + v]\theta \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now note (comes directly from the form of the Peano-Baker series):

$$e^{g^{-1}\hat{\xi}g\theta} = g^{-1}e^{\hat{\xi}\theta}g$$

$$\begin{aligned} e^{\hat{\xi}\theta} &= g\begin{pmatrix} e^{\hat{w}\theta} & [w \times (w \times v) + v]\theta \\ 0 & 1 \end{pmatrix}g^{-1} \\ &= \begin{pmatrix} I & w \times v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\hat{w}\theta} & [w \times (w \times v) + v]\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & -w \times v \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\hat{w}\theta} & [w \times (w \times v) + v]\theta + w \times \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & -w \times v \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore (assuming $\|w\| = 1$):

$$e^{\begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}\theta} = \begin{pmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})w \times v + [w \times (w \times v) + v]\theta \\ 0 & 1 \end{pmatrix}$$

which is the Rodrigues's formula for $SE(3)$.

Given $e^{At} = B$ where A, B are $n \times n$ matrices and B given, when (how) can we solve for A?

Consider $B \in S\ell(2)$, (2×2 matrices with determinant 1), when is there a corresponding matrix A? One can see that when

$$B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}$$

one cannot solve this equation (B has no real logarithm).

Theorem [Surjectivity of the exponential map $se(3) \longrightarrow SE(3)$]. Given $g \in SE(3)$, there exist $w, v, \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$ such that:

$$g = e^{\hat{\xi}\theta} \quad \text{where} \quad \hat{\xi} = \begin{pmatrix} \hat{w} & v \\ 0 & 0 \end{pmatrix}$$

proof: Let

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}, \quad R \in SO(3), \quad p \in \mathbb{R}^3$$

We ignore the trivial case in which $R = I, p = 0$ which can be solved in many different ways, one way being $\theta = 0$ and ξ arbitrary.

case i) Let $R = I, p =\neq 0$. Then we let

$$\xi = \begin{pmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{pmatrix}, \quad \theta = \|p\|$$

case ii) Let $R \neq I$. Find (w, v). We write:

$$g = \begin{pmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})w \times v + [w \times (w \times v) + v]\theta \\ 0 & 1 \end{pmatrix}$$

and solve. We have already seen there exists a unique unit vector w and $\theta \in [0, \pi]$ such that $e^{\hat{w}\theta} = R$. We only need to solve:

$$p = (I - e^{\hat{w}\theta})w \times v + [w \times (w \times v) + v]\theta$$

which is a linear equation in v. Write $w \times v = \hat{w}v, \;\; w \times (w \times v) = \hat{w}^2 v$. Then the equation may be re-written as:

$$[(I - e^{\hat{w}\theta})\hat{w} + (\hat{w}^2 + I)\theta]v = p$$

This can be solved (uniquely) for v, provided that the determinant of $[(I - e^{\hat{w}\theta})\hat{w} + (\hat{w}^2 + I)\theta] \neq 0$. Now, $(I + \hat{w}^2)w = w$ and $\hat{w}$ have a 1-D null space (since $\hat{w}w = 0$) spanned by w. Let v_1, v_2 be linearly independent vectors such that $w \cdot v_i = 0$. Without loss of generality, let v_1, v_2 be columns of $\hat{w}$. Then any $v \in \mathbb{R}^3$ may be written uniquely as:

$$v = a_1 v_1 + a_2 v_2 + a_3 w$$

Let us compute $[(I - e^{\hat{w}\theta})\hat{w} + (\hat{w}^2 + I)\theta]v$:

$$\begin{aligned} (I + \hat{w}^2)(a_1 v_1 + a_2 v_2 + a_3 w) &= a_1(I + \hat{w}^2)v_1 + a_2(I + \hat{w}^2)v_2 + a_3(I + \hat{w}^2)w \\ &= \underbrace{a_1(v_1 - v_1)}_{\text{check with any columns } of \;\hat{w}} + a_2(v_2 - v_2) + a_3 w = a_3 w \end{aligned}$$

Also

$$\begin{aligned}(I - e^{\hat{w}\theta})\hat{w}(a_1 v_1 + a_2 v_2 + a_3 w) &= \underbrace{(I - e^{\hat{w}\theta})\hat{w}(a_1 v_1 + a_2 v_2)}_{\text{since} \quad \hat{w}w=0} \\ &= (I - e^{\hat{w}\theta})(b_1 v_1 + b_2 v_2)\end{aligned}$$

where not both b_1, b_2 are equal to 0 since $\hat{w}$ is nonsingular on span $\{v_1, v_2\}$.

We know that we can change basis for the span $\{v_1, v_2\} = \hat{w}^{\perp}$, such that in the new basis $(I - e^{\hat{w}\theta})$ is represented by a 2×2 matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This is nonsingular if and only if

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \neq I \Longleftrightarrow e^{\hat{w}\theta} \neq I$$

Hence if $v = (a_1 v_1 + a_2 v_2 + a_3 w)$, with not all $a_i = 0$, then

$$[(I - e^{\hat{w}\theta})\hat{w} + (\hat{w}^2 + I)]v = b_1 v_1 + b_2 v_2 + a_3 w$$

with not all coefficients equal to 0. Q.E.D.

Next time Chasle's Theorem.