ME/SE 740

Lecture 11

Lie Groups II

Today we continue our discussion on Lie groups and Lie algebras.

Let $\mathcal{G}$ be a Lie algebra and $\alpha \subseteq \mathcal{G}$ a Lie subalgebra. We call α an <u>ideal</u> if it has the property that :

$$[A, B] \in \alpha$$

whenever $A \in \mathcal{G}$, and $B \in \alpha$. Written more simply if

$$[\mathcal{G}, \alpha] \in \alpha$$

Let G be a group and $H \subset G$ a subgroup. H is said to be a <u>normal</u> subgroup if for all $g \in G$ we have:

$$gH = Hg$$

i.e., for all $h_1 \in H$ there exists an $h_2 \in H$ such that

$$gh_1 = h_2 g$$

and such a pair exists for all $g \in G$.

Consider a Lie group G and a normal Lie subgroup H. Let h, g be the corresponding Lie algebras. One can show that in fact h is an ideal in $\mathcal{G}$. In fact, there is a $1-1$ correspondence between normal Lie subgroups and Lie algebra ideals.

Groups which have no normal subgroups other than themselves and the identity are called <u>simple</u>.

Example: $SO(n)$ for $n \geq 2$ is a simple Lie group.

$SE(n)$ is not simple. The subgroup of rigid translations is a normal subgroup.

proof: A subgroup $H \subset G$ is normal if and only if $gHg^{-1} = H$ for all $g \in G$. In the case of $SE(3)$ denote the subgroup of rigid translations by:

$$\left\{ \begin{bmatrix} I & r \\ 0 & 1 \end{bmatrix} : \quad r \in \mathbb{R}^3 \right\}$$

Let

$$\begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix}$$

be any element of $SE(3)$. Then

$$\begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}}_{h_1} \begin{pmatrix} R^T & -R^T r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & Rx + r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R^T & -R^T r \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} I & Rx \\ 0 & 1 \end{pmatrix}}_{h_2}$$

There are (4) classes of simple real Lie groups:

1. $S\ell(n, \mathbb{R})$, ($n \times n$ invertible matrices with determinant equal to 1)
2. $SO(n, \mathbb{R})$, $n =$ odd, ($n \times n$ orthogonal matrices with determinant equal to 1)
3. $SO(n, \mathbb{R})$, $n =$ even ($n \times n$ orthogonal matrices with determinant equal to 1)
4. $Sp(n)$, $n =$ even

$$(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \text{ invariant})$$

A Lie algebra $\mathcal{J}$ is said to be <u>commutative</u> or <u>Abelian</u> if for all $A, B, \in \mathcal{J}, [A, B] = 0$.

Proposition: Let $\mathcal{J}$ be a matrix Lie algebra and G the corresponding matrix Lie group. Then $\mathcal{J}$ is an Abelian Lie algebra if and only if G is an Abelian Lie group.

GROUP ACTION: Let S be a set and $(G, \dot{)}$ be a group. A left <u>group action</u> of G on S is a family of bijections $\mathcal{L}g : S \longrightarrow S$ indexed by elements of G, such that for all $g_a, g_2 \in G$:

$$\mathcal{L}g_1 \circ \mathcal{L}g_2 = \mathcal{L}g_1(\mathcal{L}g_2(s)) = \mathcal{L}g_1 g_2(s) \quad \text{for all} \quad s \in S$$

A right <u>group action</u> of G on S is a family of bijections $\mathcal{R}g : S \longrightarrow S$ indexed by elements of G, such that for all $g_a, g_2 \in G$:

$$\mathcal{R}g_1 \circ \mathcal{R}g_2 = \mathcal{R}g_2(\mathcal{R}g_1(s)) = \mathcal{R}g_2 g_1(s) \quad \text{for all} \quad s \in S$$

In short:

$$\mathcal{L}g_1 \circ \mathcal{L}g_2 = \mathcal{L}g_1 g_2(s), \quad \mathcal{R}g_1 \circ \mathcal{R}g_2 = \mathcal{R}g_2 g_1(s)$$

Examples: There are left and right group actions of $SE(m, \mathbb{R})$ on itself. Let $g \in SE(m, \mathbb{R})$:

$$g = \begin{pmatrix} R & \vec{r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L}g \left[\begin{pmatrix} U & \vec{u} \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} R & \vec{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U & \vec{u} \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{R}g \left[\begin{pmatrix} U & \vec{u} \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} U & \vec{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \vec{r} \\ 0 & 1 \end{pmatrix}$$

References:

- R. Gilmore, *Lie Groups, Lie Algebras and some of their Applications, John-Wiley 1974*

- *V. S. Varadarajan, Lie Groups, Lie Algebras and their Representations, 1984*

- *J. E. Humphreys, Inroduction to Lie Algebras and Representation Theory, Springer*

Consider the isomorphic relationship between 3-tuples and 3×3 skew symmetric matrices:

$$\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Establish a formal isomorphism:

$$\wedge \ \mathbb{R}^3 \ \longrightarrow \ so(3) \quad \text{and} \quad \vee : \ so(3) \ \longrightarrow \ \mathbb{R}^3$$

$$\hat{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^{\wedge} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Rodrigues Formula:

Let $\vec{w} \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$ and suppose that $\|\vec{w}\| = 1$. Then

$$e^{\hat{w}\theta} = I + \hat{w}\sin\theta + \hat{w}^2(1 - \cos\theta)$$

Note: This formula is easily modified in the case when $\|\vec{w}\| \neq 1$; replace $\vec{w}$ with $\frac{\vec{w}}{\|\vec{w}\|}$ and θ with $\theta\|\vec{w}\|$.

Proof:

$$e^{\hat{w}\theta} = I + \hat{w}\theta + \frac{1}{2}\hat{w}^2\theta^2 + \frac{1}{3!}\hat{w}^3\theta^3 + \frac{1}{4!}\hat{w}^4\theta^4 + \cdots$$

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \hat{w}^2 = \begin{pmatrix} -w_2^2 - w_3^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & -w_1^2 - w_3^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & -w_1^2 - w_2^2 \end{pmatrix}$$

$$\hat{w}^3 = \begin{pmatrix} 0 & \underbrace{-w_3(-w_1^2 - w_3^2) + w_2^2 w_3}_{w_3} & \underbrace{-w_2 w_3^2 + w_2(-w_1^2 - w_2^2)}_{-w_2} \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} = -\hat{w}$$

$$e^{\hat{w}\theta} = I + \hat{w}\underbrace{(\theta - \frac{1}{3!}\theta^3 + \cdots)}_{\sin\theta} + \hat{w}^2\underbrace{(\frac{1}{2}\theta^2 - \frac{1}{4!}\theta^4 + \cdots)}_{(1-\cos\theta)}$$

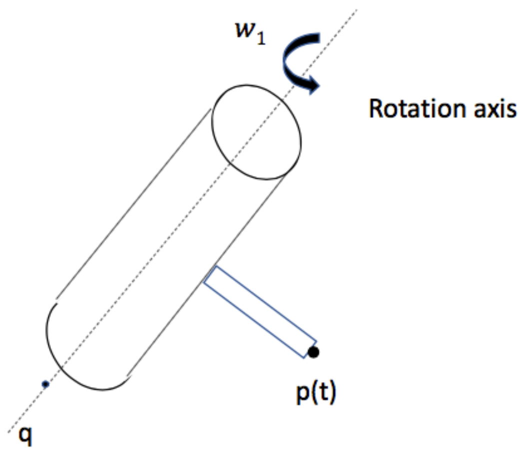

Figure 1: **Rotation about axis** w_1

Consider the rotation of a manipulator link tip about some axis w_1 (we looked at a similar rotation in Lecture 8). This motion can be described by:

$$\dot{p}(t) = w_1 \times (p(t) - q)$$

Expressing this relation in terms of 4×4 matrices in $SE(3)$ and $se3$ we can write:

$$\begin{pmatrix} \dot{p}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{w}_1 & -w_1 \times q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p(t) \\ 1 \end{pmatrix}$$

This can be solved by writing:

$$\begin{pmatrix} p(t) \\ 1 \end{pmatrix} = e^{\begin{pmatrix} \hat{w}_1 & -w_1 \times q \\ 0 & 0 \end{pmatrix} t} \begin{pmatrix} p(0) \\ 1 \end{pmatrix}$$

Similarly if a prismatic link was translating along some axis with a constant velocity v then $\dot{p}(t) = v$. Then

$$\begin{pmatrix} p(t) \\ 1 \end{pmatrix} = e^{(\hat{\xi} t)} \begin{pmatrix} p(0) \\ 1 \end{pmatrix}, \quad \text{where} \quad \hat{\xi} = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$$