Lie Groups I

Review

- Geometric Relationships in the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$
- Group Theory
- Matrix exponentials

**Definition:** A subgroup of the group of all $n \times n$ invertible matrices is called a **Lie Group** (matrix Lie Group) if it is also a closed sub-manifold (you can do calculus) on it.

**Example:** The “winding line” that is dense on the 2-torus is not a Lie Group.

**Definition:** A vector space $V$ (over $\mathbb{R}$) is a **Lie Algebra** if in addition to the vector space structure the is defined a binary operation:

satisfying the following properties:

1. **bilinearity property**
   \[ [a_1 v_1 + a_2 v_2, w] = a_1 [v_1, w] + a_2 [v_2, w] \]
   for all $v_1, v_2, w \in V$, $a_1, a_2 \in \mathbb{R}$

2. **skew symmetry**
   \[ [v, w] = -[w, v] \]
   for all $v, w \in V$

3. **Jacobi Identity**
   \[ [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0 \]
   for all $v, w, z \in V$

Three $3 \times 3$ Examples

1. **Vector cross product in $\mathbb{R}^3$**
   \[ [v, w] = v \times w \in \mathbb{R}^3 \]

2. **$so(3)$**, vector space of $3 \times 3$ skew-symmetric matrices, (with $[A, B] = AB - BA$)

3. **$sl(2)$**, set of all $2 \times 2$ matrices with trace $= 0$, (with $[A, B] = AB - BA$)

Given a matrix Lie Group $G$, we wish to study the tangent space at the identity. Let $S(t)$ be a curve in $G$ such that $S(0) = I$, $S'(0) = A$ (element in the tangent space). Let $R \in G$. Then $T(t) = RS(t)R^{-1}$ and $T'(0) = I$. Hence $T'(0) = RS'(0)R^{-1} = RAR^{-1}$ is in the tangent space at the identity.

**Proposition:** For any $R \in G$, if $A$ is in the tangent space at the identity, $T_I G$, then $\underline{RAR^{-1}}$, is also in $T_I G$.

**Proposition:** Let $R(t)$ be a curve in $G$ such that $R(0) = I$, $R'(0) = B$. Let $A$ be an element of $T_I G$. Then:

1. **$R(t)AR(t)^{-1}$** is a curve in $T_I G$

2. \[ \frac{d}{dt}|_{t=0} R(t)AR(t)^{-1} = BA - AB \]
**proof:** statement i) repeats the previous Proposition. To show statement ii) we must evaluate $\frac{d}{dt}[R^{-1}(t)]$.

Note:

\[
R(t) = R^{-1}(t) \Rightarrow R(t)R^{-1}(t) = I
\]

\[
\Rightarrow R'(0)R(0)^{-1} + R(0)\frac{d}{dt}|_{t=0}(R^{-1}(t)) = 0 = B \cdot I + I \cdot \frac{d}{dt}|_{t=0}(R^{-1}(t))
\]

\[
\Rightarrow \frac{d}{dt}|_{t=0}(R^{-1}(t)) = -R'(0) = -B
\]

The expression $BA - AB$ is known as the matrix Lie Bracket, $[B, A] = BA - AB$.

**Proposition:** Given a matrix Lie Group $G$, the tangent space at the identity $T_I G$ is a Lie Algebra with respect to this Lie Bracket.

Note: Velocities “live” in some transformed space of Lie Algebras.

**Example 1:** If $J$ is any nonsingular $n \times n$ matrix, the set of all $n \times n$ nonsingular matrices $M$ such that $M^TJM = J$ is a group (with respect to ordinary matrix multiplication).

**proof:** We will show that i) it is closed under matrix multiplication and ii) it is closed under the operation of taking inverses:

i)

\[
M_1^TJM_1 = J \quad M_2^TJM_2 = J \quad \Rightarrow (M_1M_2)^TJM_1M_2 = M_2^TJM_1M_2 = J
\]

ii)

\[
M^TJM = J, \quad \text{does this imply} \quad (M^{-1})^TJM^{-1} = J? \\
(M^TJM)M^{-1} = JM^{-1} \quad \Rightarrow M^TJM = JM^{-1} \quad \Rightarrow J = (M^T)^{-1}JM^{-1} = (M^{-1})^TJM^{-1}
\]

**SPECIAL CASE:** $J = I, \quad \Rightarrow G = O(n), \quad n \times n$ orthogonal matrices.

**Example 2:** With $n = 2m$ and

\[
J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \quad \Rightarrow G = Sp(2m), \quad \text{the symplectic group}
\]

**Example 3:**

\[
J = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow G \quad \text{the Lorentz group}
\]

Let us consider the tangent space at the identity $T_I G$. Let $R(t)$ be a curve in $G$. Then $R(t)^TJR(t) = J$ (The invariance property at the group level).

Assume $R(0) = I$, and write $R'(0) = A$. Differentiating both sides of the group invariance property at $t = 0$ we obtain:
\[
R'(0)^T J + JR'(0) = A^T J + JA = 0
\]

This is the corresponding invariance property for the Lie Algebra.

Special Case: \(J = I, A^T + A = 0.\)

Let \(\mathcal{A}\) be a set of \(n \times n\) matrices that is closed with respect to vector space operations and also with respect to the matrix Lie bracket \(A, B, \in \mathcal{A} \implies [A, B] = AB - BA \in \mathcal{A}\). In other words \(\mathcal{A}\) is a matrix Lie algebra. If \(\mathcal{A}\) is such a Lie algebra, the set of all finite products:

\[
e^{A_1} \cdot e^{A_2} \cdots e^{A_k}, \quad k \in \mathbb{Z}^+, \quad A_j \in \mathcal{A}, \quad t_j \in \mathbb{R}
\]

is the corresponding matrix Lie group.

**Example 1:** If \(\mathcal{A}\) is the Lie algebra of all \(n \times n\) matrices the corresponding Lie group is the group of \(n \times n\) invertible matrices.

**Example 2:** Let \(G = SO(3), \) (set of \(3 \times 3\) orthogonal matrices with determinant equal to 1), and \(\mathcal{A} = so(3)\) (set of \(3 \times 3\) skew symmetric matrices).

Consider the basis for \(so(3)\)

\[
\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}
\]

The Lie bracket of two of them gives the third (possible with a “-” sign) as:

\[
\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]

The group \(SO(3)\) can be thought of as all products \((t_j's \in \mathbb{R}):\)

\[
e^{0 \cdot t_1} e^{0 \cdot t_2} e^{0 \cdot t_3} = e^{0} e^{-1 \cdot t_1} e^{0} e^{0 \cdot t_2} e^{0} e^{0 \cdot t_3}
\]

One can show that:

\[
e^{0 \cdot t_1} e^{0 \cdot t_2} e^{0 \cdot t_3} = e^{0} e^{0} e^{0} = 1
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t_1 & -\sin t_1 \\
0 & \sin t_1 & \cos t_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t_1 & -\sin t_1 \\
0 & \sin t_1 & \cos t_1
\end{pmatrix}
\]

\[
+ \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1 + \frac{1}{3!} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1^2 + \frac{1}{4!} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} t_1^3 + \ldots
\]

= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t_1 & -\sin t_1 \\
0 & \sin t_1 & \cos t_1
\end{pmatrix}
\]