Lecture 9 Inhomogeneous Linear Ordinary Differential Equations*

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1 Review

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Suppose A is such that there exists a change basis P so that

$$P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

Then

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P^{-1}$$

SPECIFIC EXAMPLE:

$$A = \begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} -3/2 - \lambda & 1/2 \\ 1/2 & -3/2 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

Eigenvalues are $\lambda_1 = -1, \lambda_2 = -2$

Find the corresponding linear independent eigenvectors:

$\lambda_1 = -1$

$$\begin{pmatrix} -3/2 + 1 & 1/2 \\ 1/2 & -3/2 + 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

The eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

*This work is being done by various members of the class of 2012

$\lambda_1 = -2$

$$\begin{pmatrix} -3/2+2 & 1/2 \\ 1/2 & -3/2+2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

The eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

It's convenient to normalize these to get $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

so that $P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. Note: $P^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -3/2 & -1/2 \\ 1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \\ e^{At} &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{-t}+e^{-2t}}{2} & \frac{e^{-t}-e^{-2t}}{2} \\ \frac{e^{-t}-e^{-2t}}{2} & \frac{e^{-t}+e^{-2t}}{2} \end{pmatrix} \end{aligned}$$

2 Properties of $\Phi(t, t_0)$

1.Semi-Group Property:

$$\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$$

2.$\Phi(t_0, t_0) = I$ and for any number s,

$$\Phi(t_s, t_s) = I$$

3.For any t_0, t_1,(Follow from 1 and 2)

$$\Phi(t_0, t_1) = \Phi(t_1, t_0)^{-1}$$

EXAMPLE: A=const. coefficient (Semi-Group Property)

$$e^{At}e^{A\sigma} = e^{A(t+\sigma)}$$

What about e^{A+B} where A, B are constant $n \times n$ matrix?

$$e^{A+B} = I + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \dots$$
$$e^A e^B = (I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots)(I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \dots)$$

Compare terms:
The second order term in e^{A+B} is

$$\frac{1}{2}(A^2 + AB + BA + B^2)$$

The second order term in $e^A e^B$ is

$$\frac{1}{2}A^2 + AB + \frac{1}{2}B^2$$

These are equal $\Longleftrightarrow AB = BA$
Terms of all orders are equal $\Longleftrightarrow AB = BA$

Example of why this is useful information:
Considering

$$e^{\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} t}$$

It's not difficult to calculate:

$$\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda & \\ & 0 & \lambda \\ & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} = \begin{pmatrix} 0 & \lambda & \\ & 0 & \lambda \\ & & 0 \end{pmatrix}$$

<u>Note:</u>

$$e^{\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} t} = I + \begin{pmatrix} 0 & t & \\ & 0 & t \\ & & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & t^2 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

Nilpotent matrix A is one s.t. for some k, $A^k = 0$

$$e^{At} = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{pmatrix}$$

$$e^{\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1 \\ & & \lambda\end{pmatrix}t} = \begin{pmatrix}e^{\lambda t} & & \\ & e^{\lambda t} & \\ & & e^{\lambda t}\end{pmatrix}\begin{pmatrix}1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1\end{pmatrix}$$
$$= \begin{pmatrix}e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} \\ & & e^{\lambda t}\end{pmatrix}$$

Similarly,

$$e^{\begin{pmatrix}0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & \dots & 1\end{pmatrix}} = \begin{pmatrix}1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \dots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \frac{1}{2}t^2 & \dots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1\end{pmatrix}$$

Show that if $A = \begin{pmatrix}\sigma & 0 \\ 0 & \sigma\end{pmatrix}$, $B = \begin{pmatrix}0 & -\omega \\ \omega & 0\end{pmatrix}$, then $AB = BA$. Use this fact to compute $e^{\begin{pmatrix}\sigma & -\omega \\ \omega & \sigma\end{pmatrix}}$.

3 Inhomogeneous Linear Ordinary Differential Equations

$$\dot{x}(t) = A(t)x(t) + f(t)$$

In special case that $A(t) \equiv 0$,

$$\dot{x}(t) = f(t)$$
$$\Rightarrow x(t) = x_0 + \int_{t_0}^{t} f(s)ds$$

In the case that $A(t) \not\equiv 0$, let $\Phi(t, t_0)$ be the transition matrix associated with

$$\dot{x}(t) = A(t)x(t)$$

Define

$$z(t) = \Phi(t_0, t)x(t)$$

then,

$$\dot{z}(t) = [\frac{d}{dt}\Phi(t_0,t)]x(t) + \Phi(t_0,t)\dot{x}(t)$$

<u>Note:</u>

$$\begin{aligned}
\Phi(t_0,t)\Phi(t,t_0) &\equiv I \\
[\frac{d}{dt}\Phi(t_0,t)]\Phi(t,t_0) + \Phi(t_0,t)[\frac{d}{dt}\Phi(t,t_0)] &= 0 \\
\frac{d}{dt}[\Phi(t_0,t)] &= -\Phi(t_0,t)[\frac{d}{dt}\Phi(t,t_0)]\Phi(t_0,t) \\
&= -\Phi(t_0,t)[A(t)\Phi(t,t_0)]\Phi(t_0,t) \\
&= -\Phi(t_0,t)A(t)
\end{aligned}$$

Going back to the differential equation for $z(t)$

$$\begin{aligned}
\dot{z}(t) &= -\Phi(t_0,t)A(t)x(t) + \Phi(t_0,t)A(t)x(t) + \Phi(t_0,t)f(t) \\
&= \Phi(t_0,t)f(t)
\end{aligned}$$

$$z(t) = z_0 + \int_{t_0}^{t} \Phi(t_0,s)f(s)ds$$

$$\begin{aligned}
x(t) &= \Phi(t,t_0)z(t) \\
&= \Phi(t,t_0)z_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi(t_0,s)f(s)ds \\
&= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)f(s)ds
\end{aligned}$$

The solution to the inhomogeneous ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + f(t)$$

is given by the <u>Variation of Constants Formula</u>

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)f(s)ds$$

constant coefficients case:

$$x(t) = e^{At}x_0 + \int_{t_0}^{t} e^{A(t-s)}f(s)ds$$

<u>Example</u>: Newton's Second Law

$$\ddot{x}(t) = u(t)$$

To put this into first order form, let

$$\begin{aligned}
x_1 &= x \\
x_2 &= \dot{x}
\end{aligned}$$

Then,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$(\dot{x} = Ax + bu)$$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

By the variation of constants formula,

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & t-s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(s) ds \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} (t-s)u(s) \\ u(s) \end{pmatrix} ds \\ x(t) &= x_1(t) \\ &= x_1(0) + x_2(0)t + \int_0^t (t-s)u(s) ds \end{aligned}$$

4 Frequency domain representations

Suppose there is a constant coefficient linear system with inputs and outputs $\dot{x} = Ax + Bu$ and $y = Cx$.
A classical approach to study such a dynamic system is to take Laplace transform of it. (Recall: $\mathcal{L}\{f(t)\} = \hat{f(s)} = \int_0^\infty e^{-st} f(t) dt$)

$$\mathcal{L}\{\dot{x} = Ax + Bu\} \Leftrightarrow s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s)$$

Solving for $\hat{x}$ in the above,

$$\hat{x}(t) = (Is - A)^{-1} B\hat{u}(s)$$

Then,

$$\hat{y}(s) = C(Is - A)^{-1} B\hat{u}(s)$$

and

$$C(Is - A)^{-1} B$$

is called the transfer factor.