Homogeneous systems of linear ordinary differential equation - Lecture 8 *

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Transition matrix

$$\dot{x}(t) = A(t)x(t),\ x(t_0) = x_0$$

where $A(t)$ is a $m \times n$ matrix whose entries are continuous functions of time t.

The solution is $x(t) = \Phi(t, t_0)x_0$, where $\Phi(t, t_0)$ is the transition matrix given in terms of the *Peano-Baker series*:

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(s)\,\mathrm{d}s + \int_{t_0}^{t}\int_{t_0}^{s} A(s)A(\sigma)\,\mathrm{d}\sigma\mathrm{d}s + \dots$$

Special case $A(t) \equiv A$ is a constant $n \times n$ matrix:

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A\,\mathrm{d}s + \int_{t_0}^{t}\int_{t_0}^{s} A^2\,\mathrm{d}\sigma\mathrm{d}s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\sigma} A^3\,\mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}s + \dots$$

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\sigma} A^3\,\mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}s &= \int_{t_0}^{t}\int_{t_0}^{s} A^3(\sigma - t_0)\,\mathrm{d}\sigma\mathrm{d}s \\ &= \int_{t_0}^{t} A^3\frac{(s-t_0)^2}{2}\,\mathrm{d}s \\ &= \frac{1}{6}A^3(t-t_0)^3 = \frac{1}{3!}A^3(t-t_0)^3\end{aligned}$$

The k-th term

$$\int_{t_0}^{t}\int_{t_0}^{\sigma_{k-1}}\cdots\int_{t_0}^{\sigma_0} A^k\,\mathrm{d}\sigma_0\mathrm{d}\sigma_1\dots\mathrm{d}\sigma_{k-1} = \frac{1}{k!}A^k(t-t_0)^k$$

The Peano-Baker series can be written as:

$$\Phi(t, t_0) = I + A\cdot(t-t_0) + \frac{1}{2!}A^2(t-t_0)^2 + \frac{1}{3!}A^3(t-t_0)^3 + \dots = \sum_{k=0}^{\infty}\frac{(A\cdot(t-t_0))^k}{k!} = e^{A\cdot(t-t_0)}$$

*This work is being done by various members of the class of 2012

Scalar systems with constant coefficients

$$\dot{x} = ax,\ x(0) = x_0 \Leftrightarrow x(t) = e^{at} \cdot x_0$$

Solve this by more elementary means

$$\begin{aligned}
\dot{x} = ax \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = ax \Rightarrow \mathrm{d}x = ax\mathrm{d}t \Rightarrow \frac{\mathrm{d}x}{x} = a\mathrm{d}t \\
\Rightarrow \ln(x) = \int_0^t a\,\mathrm{d}\tau = at + C \Rightarrow x = e^{at} \cdot e^C \\
\text{and } x(0) = x_0 \Rightarrow e^C = x_0
\end{aligned}$$

For scalar systems with time dependent coefficients

$$\dot{x} = a(t)x(t),\ x(0) = x_0$$

we can write the solution

$$x(t) = e^{\int_0^t a(\sigma)\,\mathrm{d}\sigma} x_0$$

For a $n \times n$ matrix whose entries are functions of t

$$\begin{aligned}
\Phi(t, t_0) &= I + \int_{t_0}^t A(s)\,\mathrm{d}s + \int_{t_0}^t \int_{t_0}^s A(s)A(\sigma)\,\mathrm{d}\sigma\mathrm{d}s + \dots \\
&\neq e^{\int_{t_0}^t A(\sigma)\,\mathrm{d}\sigma} \text{ (in general)}
\end{aligned}$$

There is a very special case in which the matrix case has a simplication that is similar to the scalar case:

$$\dot{x} = f(t)Ax(t),\ x(0) = x_0$$

where $f(t)$ is a scalar cotinuous function and A is a constant $n \times n$ matrix.

$$\begin{aligned}
\Phi(t, t_0) &= I + \int_0^t f(\sigma)A\,\mathrm{d}\sigma + \int_0^t \int_0^{\sigma_1} f(\sigma_1)f(\sigma_2)A^2\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1 + \dots \\
&= I + \left(\int_0^t f(\sigma)\,\mathrm{d}\sigma\right) \cdot A + \left(\int_0^t \int_0^{\sigma_1} f(\sigma_1)f(\sigma_2)\,\mathrm{d}\sigma_2\mathrm{d}\sigma_1\right) A^2 + \dots
\end{aligned}$$

Let $\gamma(t) = \int_0^t f(s)\,\mathrm{d}s$, the above series can be rewritten as

$$\Phi(t, t_0) = I + \gamma(t)A + \frac{1}{2!}[\gamma(t)]^2 A^2 + \frac{1}{3!}[\gamma(t)]^3 A^3 + \dots$$

(Exercise: evaluate the k-th term in the series to show that this is correct). Seeing this pattern, we write $\Phi(t, t_0) = e^{\gamma(t)A} = e^{\left(\int_0^t f(s)\,\mathrm{d}s\right)A}$.

Specific examples

Recall the controlled pendulum system

$$\ddot{\theta} + c\dot{\theta} + \frac{g}{l}\sin(\theta) = u(t)$$

$\theta_1^* = 0$ and $\theta_2^* = \pi$ are the two equilibrium points.

Linearize about the stable equilibrium (θ_1^*) – first assuming $c = 0$ (no friction).

$$\ddot{x} + \frac{g}{l}x = \bar{u}.$$

For today's lecture we're assuming $\bar{u} = 0$ (no forcing). Thus to put the system into the framework under discussion we first orderize as follows:

$$\begin{array}{c} x_1 = \sqrt{\frac{g}{l}}x \\ x_2 = \dot{x} \end{array} \Rightarrow \begin{array}{c} \dot{x}_1 = \sqrt{\frac{g}{l}}\dot{x} = \sqrt{\frac{g}{l}}x_2 \\ \dot{x}_2 = \ddot{x} = -\frac{g}{l}x = -\sqrt{\frac{g}{l}}x_1 \end{array}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $\alpha = \sqrt{\frac{g}{l}}$.

To solve this differential equation, we compute

$$\begin{aligned}
e^{\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} t} &= I + \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}^2 t^2 + \frac{1}{3!}\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}^3 t^3 + \dots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} -\alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}^2 t^2 + \frac{1}{3!}\begin{pmatrix} 0 & -\alpha^3 \\ \alpha^3 & 0 \end{pmatrix}^3 t^3 \\
&+ \frac{1}{4!}\begin{pmatrix} \alpha^4 & 0 \\ 0 & \alpha^4 \end{pmatrix}^4 t^4 + \frac{1}{5!}\begin{pmatrix} 0 & \alpha^5 \\ -\alpha^5 & 0 \end{pmatrix}^6 t^5 + \frac{1}{6!}\begin{pmatrix} -\alpha^6 & 0 \\ 0 & \alpha^6 \end{pmatrix}^5 t^6 + \dots \\
&= \begin{pmatrix} 1 - \frac{1}{2}\alpha^2 t^2 + \frac{1}{4!}\alpha^4 t^4 - \frac{1}{6!}\alpha^6 t^6 + \dots & -\alpha t + \frac{1}{3!}\alpha^3 t^3 - \frac{1}{5!}\alpha^5 t^5 + \dots \\ \alpha t - \frac{1}{3!}\alpha^3 t^3 + \frac{1}{5!}\alpha^5 t^5 - \dots & 1 - \frac{1}{2}\alpha^2 t^2 + \frac{1}{4!}\alpha^4 t^4 - \frac{1}{6!}\alpha^6 t^6 + \dots \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}
\end{aligned}$$

Check that this satisfies the differential equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix} &= \begin{pmatrix} -\alpha\sin(\alpha t) & \alpha\cos(\alpha t) \\ -\alpha\cos(\alpha t) & -\alpha\sin(\alpha t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix} \cos(\alpha \cdot 0) & \sin(\alpha \cdot 0) \\ -\sin(\alpha \cdot 0) & \cos(\alpha \cdot 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The solution to the origina problem is

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\begin{aligned} x(t) = \sqrt{\frac{l}{g}} x_1(t) &= \sqrt{\frac{l}{g}} (\cos(\alpha t) x_1(0) + \sin(\alpha t) x_2(0)) \\ &= \cos(\alpha t) x(0) + \sqrt{\frac{l}{g}} \sin(\alpha t) \dot{x}(0) \\ &= \cos\left(\sqrt{\frac{g}{l}} t\right) x(0) + \sqrt{\frac{l}{g}} \sin\left(\sqrt{\frac{g}{l}} t\right) \dot{x}(0) \end{aligned}$$

Example 2

$$\dot{x} = Ax$$

where A is a 3×3 matrix

$$A = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix}$$

where $k_x^2 + k_y^2 + k_z^2 = 1$. A is a skew symetric matrix ($a_{ij} = -a_{ji}$).

Note: The norm of the $x(t)$ satisfying this equation is constant ($\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\| = 0$). Thus the solution "lives" on the surface of the unit sphere.

$$\begin{aligned} e^{At} = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} t + \begin{pmatrix} -k_z^2 - k_y^2 & k_y k_x & k_x k_z \\ k_y k_x & -k_z^2 - k_x^2 & k_y k_z \\ k_x k_z & k_y k_z & -k_y^2 - k_x^2 \end{pmatrix} t^2 \\ & + \begin{pmatrix} 0 & \overbrace{-k_z(-k_z^2 - k_x^2) - k_y^2 k_z}^{k_z} & \overbrace{-k_z^2 k_y - k_y(k_y^2 + k_x^2)}^{-k_y} \\ -k_z & 0 & -k_y \\ k_y & -k_x & 0 \end{pmatrix} t^3 + \ldots \\ & = I + At + \frac{1}{2} A^2 t^2 - \frac{1}{3!} A t^3 - \frac{1}{4!} A^2 t^4 + \frac{1}{5!} A t^5 \\ & = I + A\left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \ldots\right) + A^2 \left(\frac{1}{2} t^2 - \frac{1}{4!} t^4 + \frac{1}{6!} t^6 - \ldots\right) \\ & = I + A \sin(t) + A^2 (1 - \cos(t)) \end{aligned}$$

Simplications that always work

Compute $e^{\begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{pmatrix} t}$. Directly

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix} t^2 + \dots$$

(I can't see a pattern.)

Suppose I express A w.r.t. another basis $\{\vec{p}_1, \vec{p}_2\}$ and suppose that in this basis A has the form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

$$A\vec{p}_1 = \lambda_1 \vec{p}_1$$
$$A\vec{p}_2 = \lambda_2 \vec{p}_2$$

$$A \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \vec{p}_1 & \lambda_2 \vec{p}_2 \end{pmatrix} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$AP = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \Lambda$$

$$\begin{aligned} e^{At} &= e^{P\Lambda P^{-1} t} = I + P\Lambda P^{-1} t + \frac{1}{2}(P\Lambda P^{-1})(P\Lambda P^{-1}) t^2 + \dots \\ &= I + P\Lambda P^{-1} t + \frac{1}{2}(P\Lambda^2 P^{-1}) t^2 + \frac{1}{3!}(P\Lambda^3 P^{-1}) t^3 + \frac{1}{4!}(P\Lambda^4 P^{-1}) t^4 + \dots \\ &= P \left(I + \Lambda t + \frac{1}{2}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \frac{1}{4!}\Lambda^4 t^4 + \dots \right) P^{-1} \\ &= P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P^{-1} \end{aligned}$$