Dynamic Systems Theory-State-space Control - Lecture 7 *

September 25, 2012

$$x(k+1) = A(k)x(k) + Bu(k) \tag{1}$$

$$\Updownarrow$$

$$x(k) = \Phi(k,0)x(0) + \sum_{l=1}^{k} \Phi(k,l)B(l-1)u(l-1)$$

Where $\Phi(k,0)$ is the STATE TRANSTION MATRIX and $\sum_{l=1}^{k} \Phi(k,l)B(l-1)u(l-1)$ is the extensive form. This is the solution to (1).

What can be said along these lines for

$$\dot{x} = A(t)x(t) + B(t)u(t) \quad ?$$

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

Example: $\dot{x} = \sqrt{x}, \quad x(0) = 0$

solution: $x(t) = \frac{1}{4}t^2$

solution: $x(t) \equiv 0$

Solutions are not unique.

THEOREM: If A(t) is an $n \times n$ matrix, whose elements are continuous functions of time defined on the intervel $t_0 \leq t \leq t_1$, then there is at most one solution of $\dot{x} = A(t)x(t)$ which is defined on $t_0 \leq t \leq t_1$, and takes the value x_0 at $t = t_0$.

Proof: Assume, contrary to what we wish to prove, that $x_1(t)$ and $x_2(t)$ are two distinct solutions. $x_1(t_0) = x_2(t_0) = x_0$. Let $z(t) = x_1(t) - x_2(t)$. Then

$$\dot{z} = Az \qquad z(t_0) = 0$$

*This work is being done by various members of the class of 2012

$$\begin{aligned}\frac{d}{dt}\|z\|^2 &= \frac{d}{dt}(z(t)^T z(t)) = 2z(t)^T A(t) z(t)\\ &= 2\sum_{i,j=1}^{n} a_{i,j}(t) z_i(t) z_j(t)\\ &\le 2\sum_{i,j=1}^{n} \|z(t)\| \cdot \max_{k,l} |a_{kl}(t)| \|z(t)\|\\ &= 2n^2 \max_{k,l} |a_{kl}(t)| \|z(t)\|^2\end{aligned}$$

Leting $\eta(t) = 2n^2 \max_{i,j} |a_{ij}(t)|$,we have

$$\frac{d}{dt}(\|z\|^2) - \eta(t)\|z(t)\|^2 \le 0$$

Multiply both sides of the inequality by positive integrating factor

$$\rho(t) = exp \int_{t_0}^{t} -\eta(\sigma) d\sigma$$

We have

$$\begin{aligned}\frac{d}{dt}(\rho(t)\|z\|^2) &= \rho(t)\frac{d}{dt}(\|z\|^2) - \rho(t)\eta(t)\|z\|^2\\ &= \rho(t)(\frac{d}{dt}\|z\|^2 - \eta(t)\|z\|^2)\\ &\le 0\end{aligned}$$

Hence $\rho(t)\|z\|^2$ is a nonincreasing quantity, and $\rho(t_0)\|z(t_0)\|^2 \Rightarrow \rho(t)\|z\|^2 \equiv 0$ $\Rightarrow \|z\|^2 = 0 \Rightarrow z(t) \equiv 0$.

$\boxed{\text{Lipschitz continuous}}$

We wish to solve homogeneous initial problems of the form

$$\dot{x} = A(t)x(t), \qquad x(t_0) = x_0$$

Where $A(t)$ has entries that are continuous functions of time. We wish to develop the contimuous time analogue of the state transition matrix that we saw in the last lecture.

Thus the solution will be analogous to solving a finite system of algebraic equation

$$Ax = b_j$$

where

$$b_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

Where 1 is jth entry and 0's everywhere else.

Arrange the sloutions, call the j-th solution x_j, in a matrix

$$X = (\overline{x_1} \vdots \cdots \vdots \overline{x_n})$$

Then $AX = I$, so that $X = A^{-1}$. And the solution to any system

$$Az = y$$

is going to be given by

$$z = Xy$$

Our goal is to find an $n \times n$ matrix $\Phi(t)$ such that

$$\dot{\Phi(t)} = A(t)\Phi(t), \qquad \Phi(t_0) = I$$

The solution to the vector differential equation

$$\dot{x} = A(t)x(t), x(t_0) = x_0$$

will be given by

$$\dot{x}(t) = \Phi(t)x_0$$

Convergence of sequences and series

A set of functions of time $x_1(t), x_2(t), ...$ is said to be converge on an interval $[t_0, t_1]$. If there exists a function $x(t)$ such that for each t in the interval and each $\epsilon > 0$, there is a corresponding $N = N(t, \epsilon)$ such that for $n > N$,

$$|x_n(t) - x(t)| < \epsilon$$

This is sometimes called pointwise convergence.

Such a sequence is said to converge uniformly if for each $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\sup_{t_0 \le t \le t_1} |x_n(t) - x(t)| < \epsilon$$

for $n > N$.

A series $\sum_{d=1}^{\infty} x_j(t)$ is said to converge (Uniformly) if the sequence of partial sums converges (uniformly). The series is said to converge absolutely if it remains convergent when every term is replaced by its absolute value.

Given a sequence of matrices $M_1, M_2, ...$ whose elements are functions of a parameter t, the sequence converges (uniformly) of each scalar sequence $E_{ij}(M_k)$ converges (wrt. k) (uniformly), where $E_{ij}(\cdot) = ij - th$ element.

<u>THEOREM</u>: If $A(t)$ is a square matrix whose entries are continuous functions of t on a closed finite interval $[t_0, t]$, and if the sequence of matrices

$$
\begin{aligned}
M_0 &= I \\
M_k &= I + \int_{t_0}^{t} A(\sigma) M_{k-1}(\sigma) d\sigma
\end{aligned}
$$

is defined recursively, then $\{M_k(t)\}$ converges uniformly $[t_0, t_1]$. Moreover, the limit $\Phi(t, t_0) = \lim_{k\to\infty} M_k(t)$ satisfies

$$
\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0); \quad \Phi(t_0, t_0) = I
$$

<u>Proof</u>: Writing the first few terms

$$
\begin{aligned}
M_1(t) &= I + \int_{t_0}^{t} A(\sigma) d\sigma \\
M_2(t) &= I + \int_{t_0}^{t} A(\sigma) M_1(\sigma) d\sigma \\
&= I + \int_{t_0}^{t} A(\sigma) d\sigma + \int_{t_0}^{t} \int_{t_0}^{\sigma} A(\sigma) A(\tau) d\tau d\sigma \\
M_k(t) &= I + \int_{t_0}^{t} A(\sigma) d\sigma + \int_{t_0}^{t} \int_{t_0}^{\sigma} A(\sigma) A(\tau) d\tau d\sigma \\
&+ \ldots + \int_{t_0}^{t} \int_{t_0}^{\sigma_{k-1}} \ldots \int_{t_0}^{\sigma_1} A(\sigma_{k-1}) A(\sigma_{k-2}) \ldots A(\sigma_0) d\sigma_0 d\sigma_1 \ldots d\sigma_{k-1}
\end{aligned}
$$

Working on a term-by-term basis, let

$$
\eta(t) = \max_{ij} |a_{ij}(t)| \quad (a_{ij}(t) = E_{ij}(A(t))
$$

Let

$$
\gamma(t) = \int_{t_0}^{t} \eta(\sigma) d\sigma
$$

Now,

$$
\begin{aligned}
& E_{ij}(M_k(t,t_0) - M_{k-1}(t,t_0)) \\
=& E_{ij}[\int_{t_0}^{t} A(\sigma_{k-1}) \int_{t_0}^{\sigma_{k-1}} A(\sigma_{k-2}) \ldots \int_{t_0}^{\sigma_1} A(\sigma_0) d\sigma_0 d\sigma_1 \ldots d\sigma_{k-1}] \\
\leq& n^{k-1} \int_{t_0}^{t} \eta(\sigma_{k-1}) \int_{t_0}^{\sigma_{k-1}} \eta(\sigma_{k-2}) \ldots \int_{t_0}^{\sigma_1} \eta(\sigma_0) d\sigma_0 d\sigma_1 \ldots d\sigma_{k-1} \\
& (\text{Since} |E_{ij}(AB)| \leq n \cdot \max_{i,j} |E_{ij}(A)| \max_{\alpha,\beta} |E_{\alpha,\beta}(B)|) \\
=& n^{k-1} \int_{t_0}^{t} \eta(\sigma_{k-1}) \int_{t_0}^{\sigma_{k-1}} \eta(\sigma_{k-2}) \ldots \int_{t_0}^{\sigma_2} \eta(\sigma_1)\gamma(\sigma_1) d\sigma_1 \ldots d\sigma_{k-1} \\
=& n^{k-1} \int_{t_0}^{t} \eta(\sigma_{k-1}) \int_{t_0}^{\sigma_{k-1}} \eta(\sigma_{k-2}) \ldots \int_{t_0}^{\sigma_2} \frac{1}{2} \frac{d}{d\sigma_1} (\gamma(\sigma_1)^2) d\sigma_1 \ldots d\sigma_{k-1} \\
=& n^{k-1} \int_{t_0}^{t} \eta(\sigma_{k-1}) \int_{t_0}^{\sigma_{k-1}} \eta(\sigma_{k-2}) \ldots \int_{t_0}^{\sigma_3} \eta(\sigma_2) \frac{1}{2} \gamma(\sigma_2)^2 d\sigma_2 \ldots d\sigma_{k-1} \\
=& \frac{n^{k-1}\gamma(t)^k}{k!}
\end{aligned}
$$

Each term in the sum

$$E_{ij}(M_0) + \sum_{k=1}^{\infty} E_{ij}[M_k(t,t_0) - M_{k-1}(t,t_0)]$$

is less than the corresponding term in

$$
\begin{aligned}
& 1 + \gamma(t) + \frac{n\gamma(t)^2}{2!} + \frac{n^2\gamma(t)^3}{3!} + \cdots \\
=& \frac{e^{n\gamma(t)} - 1}{n} + 1
\end{aligned}
$$

So each matrix entry converges for all t by the Weierstrass M-test.(Convergence is uniform)

<u>Claim</u>: This limit is the $\Phi(t,t_0)$ that we seek.

$$
\begin{aligned}
\frac{d}{dt}(\Phi(t,t_0)) &= \frac{d}{dt}[I + \int_{t_0}^{t} A(\sigma) d\sigma + \int_{t_0}^{t} \int_{t_0}^{\sigma} A(\sigma)A(\tau) d\tau d\sigma \\
&+ \ldots + \int_{t_0}^{t} \int_{t_0}^{\sigma_{k-1}} \ldots \int_{t_0}^{\sigma_1} A(\sigma_{k-1})A(\sigma_{k-2}) \ldots A(\sigma_0) d\sigma_0 d\sigma_1 \ldots d\sigma_{k-1}] \\
&= 0 + A(t) + A(t) \int_{t_0}^{t} A(\tau) d\tau + \cdots \\
&= A(t)\Phi(t,t_0) \qquad \text{by term-by-term differentiation}
\end{aligned}
$$

Since the original series was seen to be uniformly convergent, and since the series we get by termwise differentiation is uniformly convergent, we know that the differentiated series is the true derivative of the original series. Moreover, $\Phi(t_0, t_0) = I$, so that $\Phi(t, t_0)$ is the sought after solution to the matrix differential eq.