Lecture 6 Model: Linear System with Inputs and Output in Discrete and Continuous Time

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1 Discrete Time

\[ x(k + 1) = A(k)x(k) + B(k)u(k) \quad (*) \]

\[ y(k) = C(k)x(k) \]

\[ x(k) \in \mathbb{R}^n, \text{ 'state vector'} \]

\[ u(k) \in \mathbb{R}^m, \text{ 'control or input vector'} \]

\[ y(k) \in \mathbb{R}^p, \text{ 'output vector'} \]

\[ x(k) = A(k-1)x(k-1) + B(k-1)u(k-1) \]

\[ = A(k-1)[A(k-2)x(k-2) + B(k-2)u(k-2)] + B(k-1)u(k-1) \]

\[ = A(k-1)A(k-2)x(k-2) + A(k-1)B(k-2)u(k-2) + B(k-1)u(k-1) \]

\[ \vdots \]

\[ = A(k-1)A(k-2) \cdots A(0)x(0) + A(k-1)A(k-2) \cdots A(1)B(0)u(0) + \]

\[ \cdots + B(k-1)u(k-1) \]

If we define the state transition matrix

\[ \Phi(k, l) = A(k-1)A(k-2) \cdots A(l), \]

then, the 'solution' to (*) is:

\[
  x(k) = \Phi(k, 0)x(0) + \sum_{l=1}^{k} \Phi(k, l)B(l-1)u(l-1)
\]

*This work is being done by various members of the class of 2012*
2 Continuous Time

Newton’s Law:
\[
d\frac{d^2x}{dt^2} = u(t)
\]
\[
\uparrow \text{(unit mass)}
\]

First orderize: Let \( x_1(t) = x(t), x_2(t) = \frac{dx}{dt}(t) \)

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} = \begin{pmatrix}
x_2(t) \\
u(t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u(t)
\]

The general form of state-space evolution equations for finite dimensional linear system in continuous time is:
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \\
y(t) = C(t)x(t)
\]

where, \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \).

3 Dynamic Diagrams

Basic elements are shown below:

**Summation**

**Transmission**

**Splitting**
Examples:

(i) \( x(k + 1) = a(k)x(k) + b(k)u(k) \)

(ii) Mass-spring System. (As shown in Figure 2)

\[
m\ddot{x} + kx = u(t)
\]

Think of how you would first orderize this system.

\[
\ddot{x} = \frac{1}{m}(-kx + u)
\]

\[\Rightarrow x(t) = \int \int \frac{1}{m}(-kx + u) dt dt\]

Hence, according to the expression, the dynamic diagram of the mass-spring system is shown as in Figure 3.
4 Equilibrium Points:

4.1 Discrete Time

\[ x(k + 1) = Ax(k) \]

Equilibrium equation is:

\[ x_e = Ax_e \]

if 1 is not an eigenvalue of \( A \), then the only equilibrium state is 0.

Case.

\[ x(k + 1) = Ax(k) + b \]

Equilibrium equation is:

\[ x_e = Ax_e + b \]

\[ \Rightarrow x_e = (I - A)^{-1}b \]

is the unique solution provided \( A \) does not have 1 as an eigenvalue.

4.2 Continuous Time

Case of no input/forcing \( \dot{x} = Ax \)
The equilibrium equation is:

\[ Ax = 0 \]

If 0 is not an eigenvalue of \( A \), the only equilibrium equation is \( x = 0 \).

\[ \dot{x} = Ax + b \]

The equilibrium equation is \( 0 = Ax_e + b \). If 0 is not an eigenvalue of \( A \), the equilibrium (unique) equation is:

\[ x_e = -A^{-1}b \]

### 4.3 Other Physical System That Will Be Used To Illustrate General Principles

#### 4.3.1 RLC-Circuits and Ohm’s Laws

- Voltage drop across a resistance is \( V = IR \)
- Voltage drop across a inductance is \( V = L \frac{dI}{dt} \)
- Voltage drop across a capacitance is \( V = \frac{Q}{C} \), where \( Q \) is charge on the capacitor, and \( Q(t) = \int I(s) ds \)

#### 4.3.2 Kirchoff’s Law.

The sum of voltage drops around any loop equals the applied voltage. As in the circuit in Figure 4:

\[ L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \]

This looks like a first order system, but it is actually second order in disguise since \( Q = \int I dt \). We can equivalently write:

\[ L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = E'(t) \]

We can think of this as a linear control system. It could be first orderized. We can ask questions such as how does \( I \) change as we prescribe to \( E \)?
4.3.3 The simple pendulum

At rest, the pendulum in Figure 5 is at
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix}
\]

In general, \( \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} l \sin \theta(t) \\ l \cos \theta(t) \end{pmatrix} \). The force of gravity is resolved as illustrated:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix} l \cos \theta \\ -l \sin \theta \end{pmatrix} \theta
\]

\[
\begin{pmatrix}
\ddot{x} \\
\ddot{y}
\end{pmatrix} = \begin{pmatrix} -l \sin \theta \\ -l \cos \theta \end{pmatrix} \dot{\theta}^2 + \begin{pmatrix} l \cos \theta \\ -l \sin \theta \end{pmatrix} \ddot{\theta}
\]

Gravitational force is: \( mg \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)
Acceleration force is:

\[ m \begin{pmatrix} \ddot{x} \\ \dot{y} \end{pmatrix} = -ml \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \dot{\theta}^2 + ml \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \ddot{\theta} \]

Net torque

\[ l \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \begin{pmatrix} \text{Net force vector} \\ \text{(Gravitation + acceleration)} \end{pmatrix} = l \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \left[ -ml \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \dot{\theta}^2 + ml \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \ddot{\theta} - mg \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \]

= ml^2 \ddot{\theta} + mgl \sin \theta

If there is an exogenously applied torque \( \tau \), the equation of motion is:

\[ ml^2 \ddot{\theta} + mgl \sin \theta = \tau \]

Think of \( \tau \) as a control input, determining the behavior of \( \theta(t) \) is a nonlinear control. But we will see how linear techniques can be applied.

Consider a nonlinear control system of the form introduced in the last class:

\[ \dot{x}(t) = f(x(t), u(t), t) \quad (1) \]

Think about a variational trajectory with respect to some nominal control input \( u(t) \):

\[ \frac{d}{dt} \delta x(t) = A(t) \delta x(t) + B(t) \delta u(t) \]

To first order, the variation is given by

\[ \frac{d}{dt} \delta x(t) = A(t) \delta x(t) + B(t) \delta u(t) \]

where

\[ A(t) = \left. \frac{\partial f}{\partial x} \right|_{\Phi(t; x_0, t_0, u_0(t), t)} \]

\[ B(t) = \left. \frac{\partial f}{\partial u} \right|_{\Phi(t; x_0, t_0, u_0(t), t)} \]

where \( \Phi(t; x_0, t_0, u_0(t), t) \) is the flow corresponding to the nominal control input \( u_0(\cdot) \). This is especially going to be of interest when we linearize about an equilibrium trajectory.

Returning to the pendulum example, first orderize the equation in the box:

\[ x_1 = \theta, \quad x_2 = \dot{\theta}. \]
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
\frac{x_2}{m} \\
\frac{x_2}{l} \sin x_1
\end{pmatrix} + \begin{pmatrix}
n 0 \\
1 \frac{1}{ml^2}
\end{pmatrix} \tau(t)
\]

= \begin{pmatrix}
\frac{x_2}{l} \\
\frac{x_2}{l} \sin x_1 + \tau(t) \frac{1}{ml^2}
\end{pmatrix}

Under the nominal control $\tau \equiv 0$, there are two equilibrium equations:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
\pi \\
0
\end{pmatrix}
\]

To find the variational equations, we write:

\[
\left. \begin{pmatrix}
\frac{x_2}{l} + \delta x_2 \\
-\frac{g}{l} \sin(x_1 + \delta x_1)
\end{pmatrix} \right|_{(x_1, x_2) = (0, 0)} = \begin{pmatrix}
\delta x_2 \\
-\frac{g}{l} \delta x_1
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\frac{g}{l} & 0
\end{pmatrix} \begin{pmatrix}
\delta x_1 \\
\delta x_2
\end{pmatrix}
\]

The linearized pendulum control system relative to the equilibrium equation

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

is:

\[
\begin{pmatrix}
\delta x_1 \\
\delta x_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\frac{g}{l} & 0
\end{pmatrix} \begin{pmatrix}
\delta x_1 \\
\delta x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
\frac{1}{ml^2}
\end{pmatrix} \tau(t)
\]

The pendulum system linearized about $\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
\pi \\
0
\end{pmatrix}$ is similarly show to be:

\[
\begin{pmatrix}
\delta x_1 \\
\delta x_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
\frac{g}{l} & 0
\end{pmatrix} \begin{pmatrix}
\delta x_1 \\
\delta x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
\frac{1}{ml^2}
\end{pmatrix} \tau(t)
\]