Lecture 6 Model: Linear System with Inputs and Output in Discrete and Continuous Time*

September 20, 2012

1 Discrete Time

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k) \qquad (*)\\
y(k) &= C(k)x(k)\\
x(k) &\in \mathbb{R}^n,\ 'state\ vector'\\
u(k) &\in \mathbb{R}^m,\ 'control\ or\ input\ vector'\\
y(k) &\in \mathbb{R}^p,\ 'output\ vector'
\end{aligned}
$$

$$
\begin{aligned}
x(k) &= A(k-1)x(k-1) + B(k-1)u(k-1)\\
&= A(k-1)[A(k-2)x(k-2) + B(k-2)u(k-2)] + B(k-1)u(k-1)\\
&= A(k-1)A(k-2)x(k-2) + A(k-1)B(k-2)u(k-2) + B(k-1)u(k-1)\\
&\vdots\\
&= A(k-1)A(k-2)\cdots A(0)x(0) + A(k-1)A(k-2)\cdots A(1)B(0)u(0) +\\
&\cdots + B(k-1)u(k-1)
\end{aligned}
$$

If we define the state transition matrix

$$\Phi(k,l) = A(k-1)A(k-2)\cdots A(l),$$

then, the 'solution' to (*) is:

$$\boxed{x(k) = \Phi(k,0)x(0) + \sum_{l=1}^{k} \Phi(k,l)B(l-1)u(l-1)}$$

*This work is being done by various members of the class of 2012

2 Continuous Time

Newton's Law:

$$\frac{d^2x}{dt^2} = u(t)$$
$$\uparrow \ (\textit{unit mass})$$

First orderize: Let $x_1(t) = x(t), x_2(t) = \frac{dx}{dt}(t)$

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

The general form of state-space evolution equations for finite dimensional linear system in continuous time is:

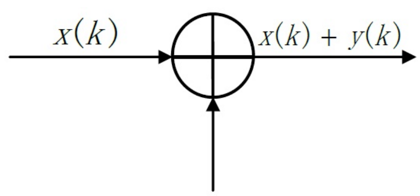
$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

where, $x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m, \ y(t) \in \mathbb{R}^p$.

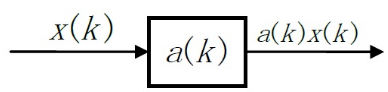
3 Dynamic Diagrams

Basic elements are shown below:

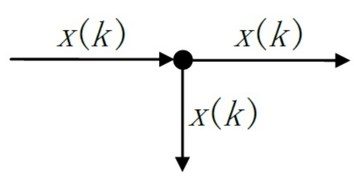
Summation



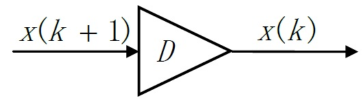
Transmission



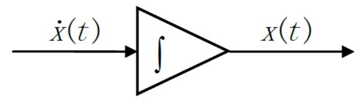
Splitting



Unit Delay

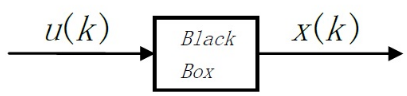


Integration



Examples:

(i) $x(k+1) = a(k)x(k) + b(k)u(k)$



Black box revealed as in Figure 1:

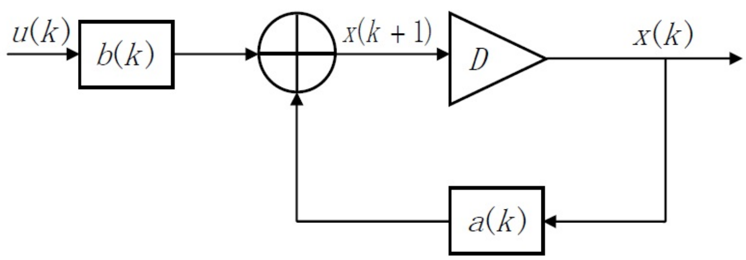


Figure 1: Black Box Revealed

(ii) Mass-spring System. (As shown in Figure 2)

$$m\ddot{x} + kx = u(t)$$

Think of how you would first orderize this system.

$$\ddot{x} = \frac{1}{m}(-kx + u)$$
$$\Rightarrow x(t) = \int\int \frac{1}{m}(-kx + u)dtdt$$

Hence, according to the expression, the dynamic diagram of the mass-spring system is shown as in Figure 3.

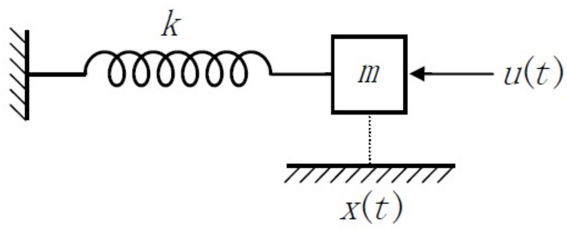


Figure 2: A Mass-Spring System

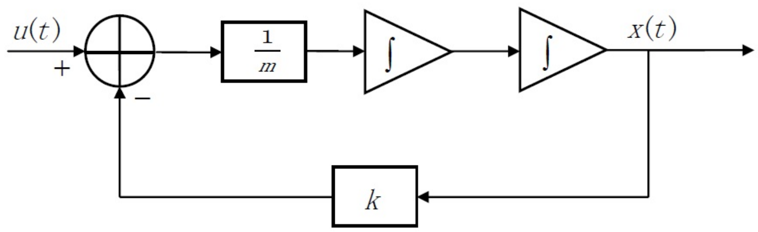


Figure 3: Dynamic Diagram for the Mass-Spring System

4 Equilibrium Points:

4.1 Discrete Time

$$x(k+1) = Ax(k)$$

. Equilibrium equation is:

$$x_e = Ax_e$$

if 1 is <u>not</u> an eigenvalue of A, then the only equilibrium state is 0.

Case.

$$x(k+1) = Ax(k) + b$$

Equilibrium equation is:

$$x_e = Ax_e + b$$
$$\Rightarrow x_e = (I - A)^{-1} b$$

is the unique solution provided A does not have 1 as an eigenvalue.

4.2 Continuous Time

Case of no input/forcing $\dot{x} = Ax$

The equilibrium equation is:

$$Ax = 0$$

If 0 is not an eigenvalue of A, the only equilibrium equation is $x = 0$.

Case.

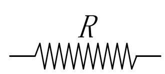
$$\dot{x} = Ax + b$$

The equilibrium equation is $0 = Ax_e + b$. If 0 is not an eigenvalue of A, the equilibrium (unique) equation is:

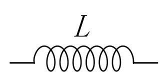
$$x_e = -A^{-1}b$$

4.3 Other Physical System That Will Be Used To Illustrate General Principles

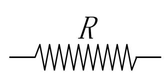
4.3.1 RLC-Circuits and Ohm's Laws



Voltage drop across a resistance is $V = IR$



Voltage drop across a inductance is $V = L\frac{dI}{dt}$



Voltage drop across a capacitance is $V = \frac{Q}{C}$, where Q is charge on the capacitor, and $Q(t) = \int^t I(s)ds$

4.3.2 Kirchoff's Law.

The sum of voltage drops around any loop equals the applied voltage. As in the circuit in Figure 4:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

This looks like a first order system, but it is actually second order in disguise since $Q = \int I dt$. We can equivalently write:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = E'(t)$$

We can think of this as a linear control system. It could be first orderized. We can ask questions such as how does I change as we prescribe to E?

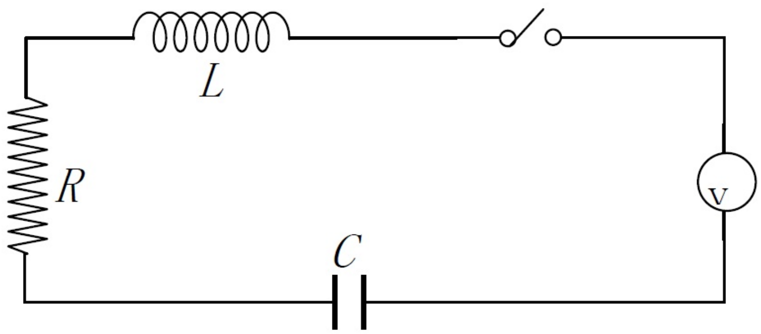


Figure 4: An RLC Circuit

4.3.3 The simple pendulum

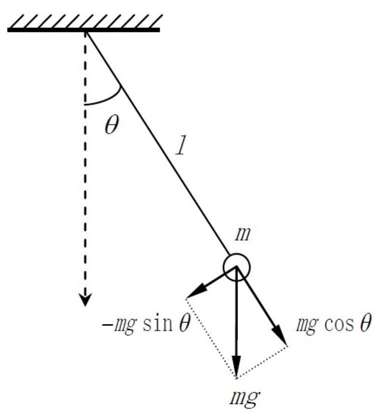


Figure 5: A Pendulum System

At rest, the pendulum in Figure 5 is at

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix}$$

In general, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} l\sin\theta(t) \\ l\cos\theta(t) \end{pmatrix}$. The force of gravity is resolved as illustrated:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} l\cos\theta \\ -l\sin\theta \end{pmatrix} \dot{\theta}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -l\sin\theta \\ -l\cos\theta \end{pmatrix} \dot{\theta}^2 + \begin{pmatrix} l\cos\theta \\ -l\sin\theta \end{pmatrix} \ddot{\theta}$$

Gravitational force is: $mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Acceleration force is: $m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = -ml\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}\dot{\theta}^2 + ml\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}\ddot{\theta}$

Net torque

$$\begin{aligned}
& l\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}\begin{pmatrix}Net\ force\ vector\\ (Gravitation\ +\ acceleration)\end{pmatrix}\\
&= l\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}\left[-ml\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}\dot{\theta}^2 + ml\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}\ddot{\theta} - mg\begin{pmatrix}0\\ 1\end{pmatrix}\right]\\
&= ml^2\ddot{\theta} + mgl\sin\theta
\end{aligned}$$

If there is an exogenously applied torque τ, the equation of motion is:

$$\boxed{ml^2\ddot{\theta} + mgl\sin\theta = \tau}$$

Think of τ as a control input, determining the behavior of $\theta(t)$ is a nonlinear control. But we will see how linear techniques can be applied.

Consider a nonlinear control system of the form introduced in the last class:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1}$$

Think about a <u>variational</u> trajectory with respect to some nominal control input $u(t)$:

$$\begin{aligned}
\frac{d}{dt}(x(t) + \delta_x(t)) &= f(x(t) + \delta_x(t), u(t) + \delta_u(t), t)\\
&= f(x(t), u(t), t) + \frac{\partial f}{\partial x}(x, u, t)\delta_x(t)\\
&+ \frac{\partial f}{\partial u}(x, u, t)\delta_u(t) + high\ order\ terms\ in\ \delta_u, \delta_x
\end{aligned}$$

To first order, the variation is given by

$$\frac{d}{dt}\delta_x(t) = A(t)\delta_x(t) + B(t)\delta_u(t)$$

where

$$\begin{aligned}
A(t) &= \left.\frac{\partial f}{\partial x}\right|_{\Phi(t;x_0,t_0,u_0(t),t)}\\
B(t) &= \left.\frac{\partial f}{\partial u}\right|_{\Phi(t;x_0,t_0,u_0(t),t)}
\end{aligned}$$

where $\Phi(t; x_0, t_0, u_0(t), t)$ is the flow corresponding to the nominal control input $u_0(\cdot)$. This is especially going to be of interest when we linearize about an equilibrium trajectory.

Returning to the pendulum example, first orderize the equation in the box: $x_1 = \theta,\ x_2 = \dot{\theta}$. Then,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\left(\dfrac{mgl}{ml^2}\right)\sin x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{ml^2} \end{pmatrix}\tau(t)$$

$$= \begin{pmatrix} x_2 \\ -\dfrac{g}{l}\sin x_1 + \dfrac{\tau(t)}{ml^2} \end{pmatrix}$$

Under the nominal control $\tau \equiv 0$, there are two equilibrium equations:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

To find the variational equations, we write:

$$\left.\begin{pmatrix} x_2 + \delta_{x_2} \\ -\dfrac{g}{l}\sin(x_1 + \delta_{x_1}) \end{pmatrix}\right|_{(x_1,x_2)=(0,0)} = \begin{pmatrix} \delta_{x_2} \\ -\dfrac{g}{l}\delta_{x_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{g}{l} & 0 \end{pmatrix}\begin{pmatrix} \delta_{x_1} \\ \delta_{x_2} \end{pmatrix}$$

The linearized pendulum control system relative to the equilibrium equation $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is:

$$\begin{pmatrix} \dot{\delta}_{x_1} \\ \dot{\delta}_{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{g}{l} & 0 \end{pmatrix}\begin{pmatrix} \delta_{x_1} \\ \delta_{x_2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{ml^2} \end{pmatrix}\tau(t)$$

The pendulum system linearized about $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ is similarly show to be:

$$\begin{pmatrix} \dot{\delta}_{x_1} \\ \dot{\delta}_{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \dfrac{g}{l} & 0 \end{pmatrix}\begin{pmatrix} \delta_{x_1} \\ \delta_{x_2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{ml^2} \end{pmatrix}\tau(t)$$