Dynamic Systems Theory - State-space Linear Systems *

September 13, 2012

Jordan Normal Form

For each λ_j, define the sequence of generalized eigenspaces

$$M^k = ker(A - \lambda I)^k$$

$$M^0 \subset M^1 \subset \ \ldots \subset M^t = M_{(\lambda)}$$

(This is called as a flag of sub-spaces)

Note: Since we are only working with one eigenvalue, we dropped the subscript.

Also define

$$W^k = (A - \lambda I)^k \mathbb{C}^n$$
$$W_{(\lambda)} = W^k$$

We choose a basis for $M_{(\lambda)} = M^t$ of the form $\{u_1, \ldots, u_{m_t}\}$ such that $\{u_1, \ldots, u_{m_k}\}$ is a basis for M^k. Other than this, there is nothing special about this basis.
We now modify the basis through a step-by-step procedure to get a representation of the desired form for the eigenvalue λ.
Let $\{u_{m_{t-1}+1}, \ldots, u_{m_t}$ be those basis elements that are in M^t but <u>not</u> in M^{t-1}. These elements do not need to be replaced but for consistency of notation, we change their names to

$$v_{m_{t-1}+1}, \ldots, v_{m_t}$$

*This work is being done by various members of the class of 2012

Now set

$$v_{m_{t-2}+\nu} = (A - \lambda I)_{v_{m_{t-1}+\nu}}$$

(recalling $(A - \lambda I)M^{k+1} \to M^k$)

Consider the set $\{u_1, \ldots, u_{m_{t-2}}\} \cup \{v_{m_{t-2}+1}, \ldots, v_{m_{t-2}+m_t-m_{t-1}}\}$
and we claim that this is a linearly independent set.

Proof of claim: If it is not linearly independent, there will be a non-trivial linear combination yielding *0*, in which atleast one of the coefficients of one of the v_i's would be non zero(this is because the set $\{u_1, \ldots, u_{m_{t-2}}\}$ is linearly independent of construction).
This means that a non-trivial linear combination of v_j's is in M^{t-2} and thus, $(A - \lambda I)^{t-2}$ would map this linear combination to zero. Then, $(A - \lambda I)^{t-1}$ would map a non-trivial linear combination of vectors in $\{v_{m_{t-1}+1}, \ldots, v_{m_t}\}$ to *0*. This non-trivial linear combination would thus be in M^{t-1}, contradicting the construction of the basis $\{u_1, \ldots, u_{m_t}\}$. This proves the claim.

Now, this linearly independent subset of M^{t-1} can be extended to form a basis $\{u_1, \ldots, u_{m_t}\} \cup \{v_{m_{t-2}+1}, \ldots, v_{m_{t-1}}\}$ for M^{t-1}.
Next set,

$$(A - \lambda I)v_{m_{t-2}+\nu} = v_{m_{t-3}+\nu} \text{ for } \nu = 1, \ldots, m_{t-1} - m_{t-2}$$

Proceed as before to obtain a new basis, $\{u_1, \ldots, u_{m_{t-3}}\} \cup \{v_{m_{t-3}+1}, \ldots, v_{m_{t-2}}\}$ of M^{t-2}.
Proceeding in this manner, we obtain a basis $\{v_1, \ldots, v_{m_t}\}$ of the entire generalized eigenspace ($M_{(\lambda)}$) such that $\{v_1, \ldots, v_{m_k}\}$ is a basis for M^k and $(A - \lambda I)v_{m_k+\nu} = v_{m_{k-1}+\nu}$ for $k \geq 1$(and $m_0 = 0$).
Another way to write this is,

$$Av_{m_k+\nu} = \lambda v_{m_k+\nu} + v_{m_{k-1}+\nu} \text{ for } k \geq 1.$$

The basis we seek is obtained by re-ordering these spaces,

$$\begin{gathered} v_1, v_{m_1+1}, v_{m_2+1}, \ldots, v_{m_{t-1}+1} \\ v_2, v_{m_1+2} \\ \vdots \\ v_{m_1}, ? \end{gathered}$$

With respect to this ordering of the basis, the matrix has the form,

$$
\le t{-}rows \begin{pmatrix} t - rows \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix} & 0 \\ 0 & 0 & \ddots \ \ddots \ \ddots \ \ddots \ \ddots \ \ddots \end{pmatrix}
$$

Each of the blocks has the eigenvalue λ on the diagonal and *1*'s on the super diagonal. The diagonal blocks are called Jordan Blocks.

Example:

$$
\left(\begin{array}{ccc|cc} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right)
$$

Suppose, a matrix with real entries has complex eigenvalues. Then you might have a Jordan normal form such as

$$
\left(\begin{array}{cccc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{array}\right)
$$

Where λ is complex $(\lambda = \alpha + \iota\beta)$ and $(\overline{\lambda} = \alpha - \iota\beta)$ is it's complex conjugate. One can do a similarity transformation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 1 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha+\iota\beta & 0 & 1 & 0 \\ 0 & \alpha-\iota\beta & 0 & 0 \\ 0 & 0 & \alpha+\iota\beta & 0 \\ 0 & 0 & 0 & \alpha+\iota\beta \end{pmatrix}$$

A matrix $\mathbf{U}$ is said to be <u>unitary</u> if $\mathbf{U}^* = \mathbf{U}^{-1}$, where $\mathbf{U}^* = \overline{\mathbf{U}^T}$. Consider the unitary matrix.

$$U = \begin{pmatrix} 1/\sqrt{2} & -\iota/\sqrt{2} \\ -\iota/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{aligned} &\begin{pmatrix} 1/\sqrt{2} & -\iota/\sqrt{2} \\ -\iota/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \alpha+\iota\beta & 0 \\ 0 & \alpha-\iota\beta \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & \iota/\sqrt{2} \\ \iota/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} (\alpha+\iota\beta)/\sqrt{2} & -(\iota\alpha-\beta)/\sqrt{2} \\ -(\iota\alpha+\beta)/\sqrt{2} & (\alpha-\iota\beta)/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & \iota/\sqrt{2} \\ \iota/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \end{aligned}$$

Similarly,

$$\left(\begin{array}{c|c} \mathbf{U} & 0 \\ \hline 0 & \mathbf{U} \end{array}\right) \begin{pmatrix} \alpha+\iota\beta & 0 & 1 & 0 \\ 0 & \alpha+\iota\beta & 0 & 1 \\ 0 & 0 & \alpha+\iota\beta & 0 \\ 0 & 0 & 0 & \alpha-\iota\beta \end{pmatrix} \left(\begin{array}{c|c} \mathbf{U}^* & 0 \\ \hline 0 & \mathbf{U}^* \end{array}\right)$$

$$= \begin{pmatrix} \alpha & -\beta & 1 & 0 \\ \beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}$$

This computation can be generalized to lead us to the notion of real Jordan Normal Form.
If $\lambda = \alpha + \iota\beta$, $\overline{\lambda} = \alpha - \iota\beta$ are complex conjugate eigenvalues of an $n \times n$ real matrix $\mathbf{A}$ and each of these eigenvalues has multiplicity S, the real Jordan form of the corresponding block is

$$\begin{pmatrix} \begin{bmatrix} \mathbf{A}_{(\alpha,\beta)} & I_2 & \dots & 0 & 0 \\ 0 & \mathbf{A}_{(\alpha,\beta)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{A}_{\alpha,\beta)} & I_2 \\ 0 & 0 & \dots & 0 & \mathbf{A}_{\alpha,\beta)} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mathbf{A}_{(\alpha,\beta)} & I_2 & \dots & 0 & 0 \\ 0 & \mathbf{A}_{(\alpha,\beta)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{A}_{\alpha,\beta)} & I_2 \\ 0 & 0 & \dots & 0 & \mathbf{A}_{\alpha,\beta)} \end{bmatrix} \end{pmatrix}$$

$$\begin{aligned} M^{k+1} &= ker(A-\lambda I)^{k+1} \\ &= \{v : (A-\lambda I)^{k+1} v = 0\} \end{aligned}$$

$M^k \quad = \{w : (A-\lambda I)^k w = 0\}$ Let $v \in M^{k+1}$.
Then, let $w = (A-\lambda I)v$

Then, $\quad (A-\lambda I)^k w = (A-\lambda I)^{k+1} v = 0$
$\Rightarrow w \in M^k$