Dynamic Systems Theory - State-space Linear Systems

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Let $w(t)$ be a sample of a Wiener process. The <u>Ito stochastic integral</u>

$$\int_a^b b(t)\,dw = \lim_{\rho\to 0}\sum_{i=1}^{n-1} b(t_i)(w(t_{i+1})-w(t_i))$$

$$\rho = \max_i |t_{i+1}-t_i|$$

<u>Properties :</u>
1. $E(\int_a^b b(t)\,dw) = 0$
2. $E(\int_a^b f(t)\,dw.\int_a^b g(t)\,dw) = q\int_a^b E(f(t)g(t))\,dt$

We write,

$$dx = f(x,t)\,dt + g(x,t)\,dw \qquad (*)$$

as a proxy for

$$\dot{x} = f(x,t) + g(x,t)\dot{w}$$

What $(*)$ really means is,

$$x_t - x_o = \int_0^t f(x,\tau)\,d\tau + \int_0^t g(x,\tau)\,dw \qquad (**)$$

$\uparrow$ This is obtained by our non-anticipating limiting process

<u>ITO's Lemma :</u> Let $x(t)$ be the unique solution of the vector Ito stochastic differential equation

$$dx = f(x,t)\,dt + g(x,t)],dw \qquad (*)$$

where x and f are n-vectors, $g(\cdot\,,\cdot)$ is an $m\times m$ matrix and $w(t)$ is an m-dimensional Wiener with

$$E(w(t)\,w(t)^T) = Q\cdot t$$

Let $Q(x,t)$ be a scalar real valued function continuously differentiable in t and having continuous second partial derivatives in x. Then the stochastic differential dQ of Q is,

$$dQ = \frac{\partial\phi}{\partial t}\,dt + \frac{\partial\phi^T}{\partial x}\,dx + \frac{1}{2}tr(gQ_g\,\frac{\partial^2\phi}{\partial x^2})\,dt,$$

where dx is defined by $(*)$, and

$$\frac{\partial^2\phi}{\partial x^2} = \begin{pmatrix} \frac{\partial\phi^2}{\partial x_1^2} & \frac{\partial\phi^2}{\partial x_1\partial x_n} & \cdots & \frac{\partial\phi^2}{\partial x_1\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi^2}{\partial x_1\partial x_n} & \frac{\partial\phi^2}{\partial x_2\partial x_n} & \cdots & \frac{\partial\phi^2}{\partial x_n^2} \end{pmatrix}$$

<u>Example :</u>

$$\dot{x} = Ax + B\dot{w} \text{ or more formally and correctly,}$$
$$dx = Ax\,dt + B\,dw$$

Compute the second moment,

$$d(x\,x^T) = x\,dx^T +\ dx\,x^T + (??)\,dt$$

The $i-j^{th}$ element of $x\,x^T$ is $x_ix_j = \phi$

The ITO correction term for this entry is,

$$\frac{1}{2}tr(BQB^T \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & 1 & \\ & 1 & & \\ 0 & \dots & & 0 \end{pmatrix})dt$$

$\uparrow$ zero's everywhere except 1's in $i-j^{th}$ & $j-i^{th}$ positions

$$= i-j^{th} \text{ element of } BQB^T\,dt$$

$$d(xx^T) = xx^TA^T\,dt + x\,dw^T\,B^T + Axx^T\,dt + B\,dw\,x^T + BQB^T\,dt$$

Taking expectations and writing $\sum_o(t) = E(x(t)x(t)^T)$

$$d\sum_o = \sum_o A^T\,dt + A\sum_o\,dt + BQB^T\,dt$$

$\Downarrow$ this is an ordinary differential equation

$$\dot{\sum}_o = A\sum_o + \sum_o A^T + BQB^T$$

This may be solved by the matrix variation of constants formula,

$$\sum_o(t) = \exp^{At}\sum_o(0)\exp^{A^Tt} + \int_0^t \exp^{A(t-\sigma)}\,BQB^T\exp^{A^T(t-\sigma)}\,d\sigma$$

Problem addressed by Kalman and Bucy was to find an optimal state estimate of

$$\begin{aligned} \dot{x} &= Ax + Bu + F\dot{w} \\ y &= Cx + \dot{v} \end{aligned}$$

Path to the solution: Find the optimal m.s.-estimate among linear observers

$$\dot{\hat{x}} = A\hat{x} + BU + K(y - C\hat{x})$$

Let $e = x - \hat{x}$

$$\begin{aligned} \dot{e} &= Ax + BU + F\dot{w} - A\hat{x} - Bu - K(y - C\hat{x}) \\ &= (A - KC)e + F\dot{w} - K\dot{v} \end{aligned}$$

$$d(ee^T) = de\, e^T + e\, de^T + L\, dt$$

where, $L =$ variance of $\xi = F\dot{w} - K\dot{v}$

$$\begin{aligned} &E((F\dot{w} - K\dot{v})(F\dot{w} - K\dot{v})^T) \\ &\quad = FQF^T + KRK^T \end{aligned}$$

where $Q = E(\dot{w}\dot{w}^T), R = E(\dot{v}\dot{v}^T)$
and we assume $\dot{w}, \dot{v}$ are uncorrelated.

$$\begin{aligned} d(ee^T) = {} & (A-KC)ee^T\, dt + ee^T(A-KC)^T\, dt + B\, dw\, e^T + e\, dw^T\, B^T + FQF^T\, dt + \\ & KRK^T\, dt \end{aligned}$$

Taking expectations of both sides, writing $\sum_{ee}(t) = E(ee^T)$

$$\tfrac{d}{dt} \textstyle\sum_{ee}(t) = A_K \sum_{ee}(t) + \sum_{ee}(t) A_K^T + FQF^T + KRK^T$$

where $A_K = A - KC$. We wish to find the gain matrix K that minimizes the m.s.-error $\sum_{ee}(t)$ at each time t_1.

What it means for one symmetric matrix to be less than another:

$$P \le Q \Leftrightarrow x^T P x \le x^T Q x \qquad \text{for all } x.$$

Let $\sum_{ee}^o(t)$ be the optimal error variance. Then writing

$P = \sum_{ee}(t) - \sum_{ee}^o(t)$ where $\sum_{ee}(t)$ error for a nominal gain $K(t)$.

The P satisfies

$$\dot{P} = \dot{\textstyle\sum} - \dot{\textstyle\sum}^o$$

$$= (A - KC) \textstyle\sum^o + \sum(A - KC)^T + FQF^T + KRK^T - (A - K^oC) \sum^o -$$

$$\sum(A - K^oC)^T - FQF^T - KRK^{o^T}$$

$$= (A-KC)P+P(A-KC)^T+(K^o-K)R(K^o-K)^T+(K^o-K)(C\sum^o -RK^{o^T})+ \\ (\sum_o C^T - K^oR)(K^o - K)^T$$

$= (A - KC)P + P(A - KC)^T + L$
where $L = (K^o - K)R(K^o - K)^T + (K^o - K)(C\sum^o -RK^{o^T})+$
$(\sum^o C^T - K^oR)(K^o - K)^T$

The solution to this equation is

$$P(t) = \Phi_{A_C}(t, 0)P_0\Phi_{A_C}(t, 0)^T + \int_\delta^t \Phi_{A_C}(t, s)L\Phi_{A_C}(t, s)^T\, ds$$

We assume $P_0 = \sum_{ee}(0) - \sum_{oe}^o(0) = 0$.

$P(t)$ will be positive semi-definite precisely when L is positive semi-definite. In rder for L to be positive semi-definite for choices of nominal gain K, we must have,

$$\sum^o C^T - K^oR = 0$$

Thus the optimal gain is

$$K^o = \sum_{ee}^o(t)C^TR^{-1}$$

Let's verify that this 'Kalman gain' is optimal.
First note $\sum^o(t)$ satisfies

$$\dot{\sum^o} = A_K\sum^o + \sum^o A_K^T + FQF^T + K^oRK^{o^T}$$
$$\text{where } A_K = A - K^oC, K^o = \sum^o C^TR^{-1}$$

$$\dot{\sum^o} = (A - \sum^o C^TR^{-1}C)\sum^o + \sum^o(A^T - C^TR^{-1}C\sum^o)+ \\ FQF^T + \sum^o C^TR^{-1}C\sum^o$$

$$= A\sum^o + \sum^o A^T + FQF^T - \sum^o C^TR^{-1}C\sum^o$$

Using this equation for the optimal error we rewrite the equation for
$P = \sum_{ee} - \sum_{ee}^o$

$$\dot{P} = A_)KC)\sum_{ee} + \sum_{ee}(A - KC)^T + FQF^T + KRK^T - S\sum^o - \sum^o A^T \\ -FQF^T + \sum^o C^TR^{-1}C\sum^o$$

$$= AP + PA^T + KRK_K^TC\sum^o - \sum^o C^TK^T + \sum^o$$

$$= AP + PA^T + \underbrace{(K - \sum^o C^TR^{-1})R(K - \sum C^TR{-}1)^T}_{L}$$

$P(t) = \int_0^T \Phi_A(t,s)L(s)\Phi_A^T(t,s)\,ds$

This is positive semi definite and $= 0 \Leftrightarrow$

$$K - \sum^o C^T R^{-1} = 0 \qquad \text{This shows}$$
$$\sum_{ee}(t) \geq \sum^o(t)$$

$$\text{with equality holding} \Leftrightarrow \quad K = \sum^o C^T R^{-1}.$$

<u>SUMMARY :</u> Given the finite dimensional linear system,

$$\dot{x} = Ax + Bu + F\dot{w}$$

$$y = Cx + v$$

$\dot{w}$ is Gaussian white noise $N(O,Q)$
$\dot{v}$ is Gaussian white noise $N(O,R)$
$\dot{w}, \dot{v}$ uncorrelated

The mean-squared optimal observer is

$$\dot{\hat{x}} = (A - \sum^o(t)C^T R^{-1}C)\hat{x} + Bu + \sum^o C^T R^{-1}y$$
$$\hat{x}(0) = E(x_0) = \bar{x_0}$$

with $\sum^o(t)$ given as the solution of

$$\dot{\sum}^o(t) = A\sum^o + \sum^o A^T + FQF^T - \sum^o c^T R^{-1}C\sum^o$$

$$\sum^o(0) = E((x_0 - \bar{x_0})(x_0 - \bar{x_0})^T)$$