Dynamic Systems - State Space Control - Lecture 23 *

December 4, 2012

White Noise and Wiener Processes

Think of a semi - infinite bar Let $u(x,t)$ denote the temperature in the bar at x at time t.

Rate of heat transfer from left to right accross a section at $x = x_0$ is,

$$\begin{aligned} H(x_0,t) &= -\lim_{d \to 0} K.A \frac{u(x_0 + \frac{d}{2}, t) - u(x_0 - \frac{d}{2}, t)}{d} \\ &= -K.A \frac{\partial u(x_0,t)}{\partial x} \end{aligned}$$

The net rate at which heat flows in a segment between x_0 and $x + \Delta x$ is,

$$H(x_0,t) - H(x_0 + \Delta x, t) = K.A(\frac{\partial u(x_0 + \Delta x, t)}{\partial x} \frac{\partial u(x_0,t)}{\partial x})$$

Denote this by Q. Average change in temperature during the time period Δt os proportional to $Q.\Delta t$ and inversly proportional to the mass Δm of the infinitessimal segment $= \rho A \Delta x$

This amount is $\frac{Q.\Delta t}{s \rho A \Delta x}$,

Where 's' is a proportionality constant which is equal to the specific heat of the material.

The average temperature at time t will be the actual temperature at some point $x_0 + \theta(t).\Delta x$, where $0 \leq \theta(t) \leq$)

*This work is being done by various members of the class of 2012

$$u(x_0 + \theta(t).\Delta x, t + \Delta t) - u(x_0 + \theta(t).\Delta x, t) = \frac{Q.\Delta t}{s\rho A \Delta x}$$

Dividing throught by Δt and letting $\Delta t \to 0.\Delta x \to 0$ we obtain:

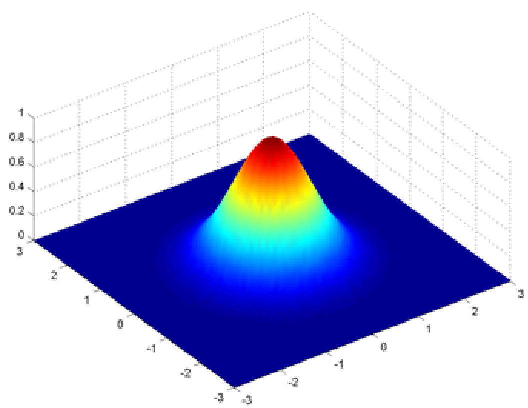
$$\frac{\partial u(x_0, t)}{\partial x} = \frac{\alpha^2}{2} . \frac{\partial^2 u(x_0, t)}{\partial x^2}$$

Where all the physical constants are lumped together in α.

Next, notice that $p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}$ (x, t are variables) satisfies the equation,

$$\frac{\partial p}{\partial t} = \frac{\alpha^2}{2} . \frac{\partial^2 p}{\partial x^2}$$

$$p(x, 0) = \delta(x)$$



How might a stochastic process associated with $p(x,t)$ come up?
The simplest stochastic process is **random walk** in 1 dimension.

$$x_0 = 0$$
$$x_{n+1} = x_n + z_n$$
$$p\{z_n = 1\} = 1/2, \quad p\{z_n = -1\} = 1/2$$

Suppose z_n obeys porbability, $Pr\{z_n = \Delta\} = 1/2, \quad Pr\{z_n = -\Delta\} = 1/2$

Consider this random walk as $n \to \infty, \Delta \to 0$ and $T =$ time interval between steps $\to 0$ such that,

$$nT \sim t$$

$$\Delta \sim \sqrt{T}$$

$\frac{\Delta}{T}$= step size per unit time (= Velocity) $\sim \frac{\sqrt{T}}{T} = \frac{1}{\sqrt{T}} \to \infty$

$Pr\{x(t) \in [x + dx]\} \sim \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$

This is called **unit variance wiener process**.

For the random walk:

$$E(x(n\tau)) = 0$$
$$E(x(n\tau)^2) = ns^2$$

s^2 is the same variance parameter $= n\tau \to t$
The wiener process is an <u>independent increments</u> process:
For $t_2 > t_1$ $x(t_2) - x(t_1)$:

$$E\{(x(t_2) - x(t_1))x(t_1)\} = 0$$

$$E(x(t_1)x(t_2)) = E(x(t_1)^2) \text{ if } t_2 > t_1 = t_1$$

$$R_{xx}(t_1, t_2) = E(x(t_1)x(t_2)) = min(t_1, t_2)$$

A slightly more general treatment has,

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}; \quad p(x,0) = \delta(x)$$

The first and second order statistical characteristics are,

- $E(x(t)) = 0 \qquad \left(= \int_{-\infty}^{\infty} x\, p(x,t)\, \mathrm{d}x \right)$
- $E(x(t)^2) = \sigma^2 t \qquad \left(= \int_{-\infty}^{\infty} x^2\, p(x,t)\, \mathrm{d}x \right)$
- $R_{xx}(t_1, t_2) = \sigma^2 min(t_1, t_2) \qquad \left(= E(x(t_1)x(t_2)) \right)$

DEFINITION: A stochastic process $x(t)$ is said to be white noise if it is stationary and

$$R_{xx}(\tau) = E(x(t)x(t+\tau) = q\delta(\tau)$$

Suppose $x(t)$ is white noise and,

$$\frac{(d)y}{(d)t} = x(t), \quad y(0) = 0$$

$$y(t) = \int_0^t x(\tau)\,\mathrm{d}\tau$$

Formally,

$$\begin{aligned} E_{yy}(y(t_1)y(t_2)) &= E\Big[\int_0^{t_1}\int_0^{t_2} x(\tau)x(\sigma)\,\mathrm{d}\sigma\mathrm{d}\tau\Big] \\ &= \int_0^{t_1}\int_0^{t_2} E\big(x(\tau)x(\sigma)\big)\,\mathrm{d}\sigma\mathrm{d}\tau \\ &= \int_0^{t_1}\int_0^{t_2} q\delta(\tau-\sigma)\,\mathrm{d}\sigma\mathrm{d}\tau \end{aligned}$$

If $t_2 > t_1$ this is,

$$= \int_0^{t_1} q\,\mathrm{d}\tau = qt_1$$

(Conversly, if $t_1 > t_2$ it is $= qt\dot{z}$)

$$\begin{aligned} &E(y(t)) = 0 \\ &E(y(t_1)y(t_2)) = q\,min(t_1, t_2) \end{aligned}$$

Formally, we've shown that the derivative of a wiener process is white noise.

Consider a linear system driven by a stochastic input,

$$\dot{y} = Ay + bx$$

We can easily write down the evolution of the <u>means</u>,

$$\dot{\bar{y}} = A\bar{y} + b\bar{x}$$

Lets compute the second order statistics,

$$R_{yy}(t_1, t_2) = E(y(t_1)y(t_2)^T)$$

$$\begin{aligned}
\frac{\partial}{\partial t_2}(R_{yy}(t_1, t_2)) &= E(y(t_1)y(t_2)^T) \\
&= E(y(t_1)^T A^T + b^T x(t_2)) \\
&= E(y(t_1)y(t_1)^T)A^T + E(y(t_1)x(t_2))b^T \\
&= R_{yy}(t_1, t_2)A^T + R_{yx}(t_1, t_2)b^T
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2} &= \frac{\partial}{\partial t_1}\Big\{E(y(t_1)y(t_1)^T)A^T + E(y(t_1)x(t_2))b^T\Big\} \\
&= E[(Ay(t_1) + bx(t_1))y(t_2)^T]A^T + E[(Ay(t_1) + bx(t_1)x(t_2)]b^T \\
&= AR_{yy}(t_1, t_2)A^T + bR_{xy}(t_1, t_2)A^T + AR_{yx}(t_1, t_2)b^T + bR_{xx}(t_1, t_2)b^T
\end{aligned}$$

Compute the mean and covariance of $y(t)$ assuming that $y(0) = 0$

$$\begin{aligned}
R_{yy}(0, t_2) &= E(y(0)y(t_2)^T) = 0 \\
R_{yy}(t_1, t_2) &= e^{At}R_{yy}(0, t_2) + \int_0^t e^{A(t-s)}bR_{xy}(s, t)\mathrm{d}s
\end{aligned}$$

What about $R_{xy}(t_1, t_2)$?

$$\begin{aligned}
\frac{\partial}{\partial t_2}R_{xy}(t_1, t_2) &= \frac{\partial}{\partial t_2}E\big[x(t_1)y(t_2)^T\big] \\
&= E\big[x(t_1)y(t_2)A^T + x(t_2)b^T\big] \\
&= R_{xy}(t_1, t_2)A^T + R_{xx}(t_1, t_2)b^T
\end{aligned}$$

$$R_{xy}(t_1, t_2)^T = e^{At}R_{xy}(t_1, 0) + \int_0^t e^{A(t-\sigma)}bR_{xx}(t-\sigma)\,\mathrm{d}\sigma, \quad e^{At}R_{xy}(t_1, 0) = 0$$

$$\implies R_{xy}(t_1, t_2) = \int_0^{t_2} b^T e^{A(t_2-\sigma)}R_{xx}(t-\sigma)\,\mathrm{d}\sigma$$

Finally we go back to the expression for R_{yy},

$$\begin{aligned}
R_{yy}(t_1, t_2) &= \int_0^{t_1} e^{t_1-\tau} b R_{xy}(\tau, t_2)\, \mathrm{d}\tau \\
&= \int_0^{t_1} e^{A(t_1-\tau)} b \int_0^{t_2} b^T e^{A^T(t_2-\sigma)} R_{xx(\tau,\sigma)}\, \mathrm{d}\sigma\, \mathrm{d}\tau \\
&= \int_0^{t_1}\int_0^{t_2} e^{A(t_1-\tau)} b b^T e^{A(t_2-\tau)} R_{xx}(\tau, \sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau
\end{aligned}$$

Special case: $x(t) =$ White noise

$$R_{xx}(t_1, t_2) = q.\delta(t_1 - t_2)$$

$$\begin{aligned}
R_{yy} &= \int_0^{t_1}\int_0^{t_2} e^{A(t_1-\tau)} b b^T e^{A(t_2-\tau)} q.\delta(t-\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau \\
&= \int_0^{min(t_1,t_2)} e^{A(t_1-\tau)} b b^T e^{A(t_2-\tau)} q\, \mathrm{d}\tau \qquad (*)
\end{aligned}$$

The risks associated with blindly applying calculus rules when white noise is involved.

Consider, $\dot{x} = Ax + b\dot{w}$

$$\begin{aligned}
\frac{d}{dt}(xx^T) &= (Ax + b\dot{w})x^T + x(x^T A + \dot{w} b^T) \\
&= Axx^T + xx^T A^T + b\dot{w}x^T + x\dot{w}b^T
\end{aligned}$$

Taking expectations of both sides,

$$\frac{d}{dt}E(xx^T) = AE(xx^T) + E(xx^T)A^T + bE(\dot{w}x^T) + E(x\dot{w})b^T$$

Being able to solve this hinges on knowing $E(x\dot{w})$,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} b\dot{w}(s)\, \mathrm{d}s$$

Assume $e^{At}x_0 = 0$

$$\begin{aligned} E\big(x(t)\dot{w}(t)\big) &= \int_0^t e^{A(t-s)} b E(\dot{w}(s)\dot{w}(t))\,\mathrm{d}s \\ &= \int_0^t e^{A(t-s)} b q \delta(t-s)\,\mathrm{d}s \\ &= bq \end{aligned}$$

Then letting $E(xx^T)$ be denoted by $\sigma_x(t)$

$$\sigma_x(t) = e^{At}\sigma_x(0)e^{A^T t} + 2\int_0^t e^{A(t-\sigma)} b b^T e^{A^T(t-\sigma)}\,\mathrm{d}\sigma \qquad (*)$$

$(*) \neq (**)$
The resolution to this situation is to carefully redevelop calculus - we'll follow Ito.

Suppose we wish to study the evolution of a wiener process on an interval $a \leq t \leq b$.Partition the interval $a = t_0 < t_1 < \cdots < t_n = b$ and define the ITo stochastic integral,

$$\lim_{||t_{i+1-t_i} \to 0} \sum_{i=0}^{n-1} b(t_i)\big[w(t_{i+1}) - w(t_i)\big] = \int_a^b b(t)\mathrm{d}\dot{w}$$