Least Squares Optimization - Lecture 21 *

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Least Squares Optimization

Theorem 1. *Suppose (A, B) is a controllable pair, then the control that steers*

$$\dot{x} = Ax + Bu, \; x(0) = x_0$$

so as to minimise (the Matrix Riccati equation)

$$\eta = \int_0^T x^T Q x + u^T R u \; \mathrm{dt}$$

(Q positive semi-definite and R positive definite)
is given by

$$u(t) = -R^{-1} B^T M(t) x(t)$$

where M satisfies

$$\dot{M} = -A^T M - MA + MBR^{-1}B^T M - Q, M(T) = 0$$

The minimum value of η is $\eta_0 = x^T(0)M(0)x(0)$ (which can not be changed with a control input).

Proof. Assume we have a solution to the Riccati equation that governs $M(\cdot)$.

*This work is being done by various members of the class of 2012

Write

$$
\begin{aligned}
\eta &= \int_0^T x^T Q x + u^T R u \, \mathrm{d}t \\
&= \int_0^T -x^T \dot{M} x - x^T A^T M x - x^T M A x + x^T M B R^{-1} B^T M x + u^T R u \, \mathrm{d}t \\
&= \int_0^T -x^T \dot{M} x - \dot{x}^T \dot{M} x + x^T A^T M x + u^T B^T M x - x^T M \dot{x} + x^T M A x \\
&\quad + x^T M B u - x^T A^T M x - x^T M A x + x^T M B R^{-1} B^T M x + u^T R u \, \mathrm{d}t \\
&= \int_0^T -\frac{\mathrm{d}}{\mathrm{d}t}(x^T M x) + u^T B^T M x + x^T M B u + x^T M B R^{-1} B^T M x + u^T R u \, \mathrm{d}t \\
&= x^T(0) M(0) x(0) + \int_0^T (u + R^{-1} B^T M x)^T R (u + R^{-1} B^T M x) \, \mathrm{d}t
\end{aligned}
$$

Because R is positive-definite the last (integral) term is ≥ 0. We have equality $\Leftrightarrow u(t) = -R^{-1} B^T M(t) x(t)$. □

Theorem 2. *Let W be the controllability Grammanian for the system*

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$

If $u_0(\cdot)$ is any control of the form

$$u_0(t) = -B\Phi(t, t_0)\xi$$

where ξ satisfies

$$W(t_0, t_1)\xi = x_0 - \Phi(t_0, t_1) x_1$$

then the control steers the system from x_0 at $t = t_0$ to x_1 at $t = t_1$, and if $u_1(\cdot)$ is any other control that steers the system from x_0 at t_0 to x_1 at t_1, then

$$\int_{t_0}^{t_1} \|u_1(t)\|^2 \, \mathrm{d}t \geq \int_{t_0}^{t_1} \|u_0(t)\|^2 \, \mathrm{d}t$$

Proof.

$$
\begin{aligned}
x_1 &= \Phi(t_1, t_0) \left(x_0 \int_{t_0}^{t_1} \Phi(t_0, s) B(s) u_1(s) \, \mathrm{d}s \right) \\
&= \Phi(t_1, t_0) \left(x_0 \int_{t_0}^{t_1} \Phi(t_0, s) B(s) u_0(s) \, \mathrm{d}s \right)
\end{aligned}
$$

Substracting

$$\int_{t_0}^{t_1} \Phi(t_0, s) B(s) (u_1(s) - u_0(s)) \, \mathrm{d}s = 0$$

Premultiplication by ξ^T together with the definition of $u_0(\cdot)$ gives

$$\begin{aligned}
&\int_{t_0}^{t_1} u_0^T(t)(u_1(t) - u_0(t))\ \mathrm{d}t = 0 \\
&\int_{t_0}^{t_1} \|u_1(t)\|^2 - \|u_0(t)\|^2\ \mathrm{d}t = \int_{t_0}^{t_1} u_1^T(t)u_1(t) - u_0^T(t)u_0(t)\ \mathrm{d}t \\
&= \int_{t_0}^{t_1} u_1^T(t)u_1(t) - u_0^T(t)u_0(t) - 2u_0^T(t)u_1(t) + 2u_0^T(t)u_0(t)\ \mathrm{d}t \\
&= \int_{t_0}^{t_1} u_1^T(t)u_1(t) - 2u_1^T(t)u_0(t) + u_0^T(t)u_0(t)\ \mathrm{d}t \\
&= \int_{t_0}^{t_1} (u_1(t) - u_0(t))^T(u_1(t) - u_0(t))\ \mathrm{d}t \\
&= \int_{t_0}^{t_1} \|u_1(t) - u_0(t)\|^2\ \mathrm{d}t \geq 0
\end{aligned}$$

And there is equality $\Leftrightarrow u_1(t) = u_0(t)$. □

$$\eta = \int_0^T l(x,u)\ \mathrm{d}(t) + \Psi(x(T))$$

Theorem 3 (Endpoint value penalized)**.** *Let A, B, Q, Q_T be matrices where both Q and Q_T are $n \times n$ symmetric and positive semi-definite. Let $M(\cdot)$ be the unique $n \times n$ positive semi-definite solution to*

$$\dot{M} = -A^T M - MA + MBB^T M - Q, M(T) = Q_T$$

Then there exists a control $u(\cdot)$ that steers the system

$$\dot{x} = Ax + Bu,\ x(0) = x_0$$

and minimized the performance metric

$$\eta = \int_0^T x^T Q x + \|u(t)\|^2\ \mathrm{d}t + x^T(T)Q_T x(T)$$

The minimum value of η is $x^T(0)M(0)x(0)$, and the minimizing control is

$$u(t) = -B^T M(t)\Phi(t,0)x_0$$

where $\Phi(t,t_0)$ is the transition matrix associated with

$$\dot{x} = (A - BB^T M(t))x$$

One can let $T \to \infty$ and the form of the Riccati equation is such that we need to consider a steady-state solution K

$$A^T K + KA - KBB^T K + Q = 0$$

If (A, B) is a controllable pair, a unique positive semi-definite solution exists. Moreover, the matrix $A - BB^T K$ has all eigenvalues in the left half-plane.

With $T \to \infty$, this linear quadratic optimization problem, becomes the *quadratic regulator problem (LQR problem).*

1 Fixed endpoint problems

Consider

$$\dot{x} = Ax + Bu, \ x(0) = x_0$$

$$\eta = \int_0^T x^T(s)Qx(s) + u^T(s)u(s)\mathrm{d}s$$

Assume there is a solution to the Riccati equation

$$\dot{M} = -A^T M - MA + MBB^T M - Q, M(t_1) = M_1$$

on the interval $[t_0, t_1]$.

Theorem 4. *A trajectory $x(t)$ minimizes η for the given system and boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1 \Leftrightarrow$ it minimizes*

$$\bar{\eta} = \int_{t_0}^{t_1} \|v(t)\|^2 \ \mathrm{d}t$$

for the differential equation

$$\dot{x} = (A - BB^T M(t))x(t) + Bv(t)$$

with the same boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$.

Proof. Assume the system

$$\dot{x} = Ax(t) + Bu(t)$$

is driven from x_0 at $t = t_0$ to x_1 at $t = t_1$ by a control of the form

$$u(t) = -B^T M(t)x(t) + v(t)$$

where $M(\cdot)$ is the solution of the Riccati equation given above.

The "cost" of the control is

$$\begin{aligned}
\eta &= \int_{t_0}^{t_1} x^T(t)Qx(t) + u^T(t)u(t) \text{ d}t \\
&= \int_{t_0}^{t_1} -x^T\dot{M}x - x^TA^TMx - x^TMAx + x^TMBB^TMx \\
&+(v^T - x^TMB)(v - B^TMx) \text{ d}t \\
&= \int_{t_0}^{t_1} -x^T\dot{M}x - x^TA^TMx - x^TMAx + x^TMBB^TMx \\
&+v^Tv - x^TMBv - v^TB^TMx + x^TMBB^TMx \text{ d}t \\
&= \int_{t_0}^{t_1} -x^T\dot{M}x - x^T(A - MBB^T)x - x^TM(A - BB^TM)x \\
&-V^TB^TMx - x^TMBv + v^Tv \text{ d}t \\
&= \int_{t_0}^{t_1} -x^T\dot{M}x - \dot{x}^TMx - x^TM\dot{x} + v^Tv \text{ d}t \\
&= -x^T(t_1)M(t_1)x(t_1) + x^T(t_0)M(t_0)x(t_0) + \int_0^T v^Tv \text{ d}t
\end{aligned}$$

The first two quantities in the last relation are fixed. The only term that can be adjusted is the integral term $\int_0^T v^Tv \text{ d}t$.

This can be minimized subject to the differential equation

$$\dot{x} = (A - BB^TM(t))x(t) + Bv(t)$$

together with the boundary conditions

$$x(t_0) = x_0, x(t_1) = x_1$$

□

This yields an algorithm for finding the fixed endpoint control:

1. Solve the Riccati equation for $M(\cdot)$.
2. Use the procedure outlined at the beginning of the class to find v_{opt} which steers
$$\dot{x} = (A - BB^TM(t))x(t) + Bv(t)$$
from $x(t_0) = x_0$ to $x(t_1) = x_1$ so as to minimize the integral
$$\int_{t_0}^{t_1} \|v(t)\|^2 \text{ d}t$$
3. The control for the original problem is
$$u(t) = -B^TM(t)x(t) + v_{opt}(t)$$

2 Summary

The PMP yields necessary conditions for optimal control problems of the form

Find $u(\cdot)$ that steers the system

$$\dot{x} = f(x, u)$$

so as to minimize

$$\eta = \int_{t_0}^{t_1} l(x, u)\ \mathrm{d}t + \Psi(x(t_1))$$

PMP solution: there is $\lambda(t)$ such that

$$\dot{x} = f(x(t), u(t)), x(t_0) = x_0$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda(t) - \frac{\partial l}{\partial x}$$

For components of the state that are not specified at $t = t_1$, the corresponding components of $\lambda(t_1)$ and $\frac{\partial \Psi}{\partial x}(x(t_1))$ are equal.

$$\frac{\partial l}{\partial u} + \frac{\partial}{\partial u}(\lambda^T f(x, u)) \equiv 0$$

Very important special cases:

1. $l(x, u) \equiv 1$ corresponds to the time optimal problem (not covered).
2. $l(x, u) = x^T QX + u^T Ru$ and $f(x, u) = Ax + Bu$ corresponds to the problem of linear-quadratic optimization.