Lecture 20 *

December 5, 2012

1 Pontryagin Maximum Principal (First Version)

Theorem: Pontryagin Maximum Principal-First Version suppose it is desired to steer the sate of

$$\dot{x} = f(x,u), x(0) = x_0$$

so as to maximize (minimize)

$$J = \int_0^T l(x,u)dt + \psi(x(T)))dt$$

If this defines a normal problem then there exists a *costate* vector $\lambda(t)$ such that λ, together with the optimizing values of x,u jointly satisfy

$$\dot{x} = \frac{\delta H}{\delta \lambda}$$

$$\lambda(T) = \frac{\delta \psi}{\delta x}(x(t))$$

$$\frac{\delta H}{\delta u} = 0$$

Where the Hamiltonian H is defined as

$$H(\lambda, X, u) = \lambda^T F(x,u) + l(x,u)$$

Consider the same problem, optimization, in which the terminal value of x(T) is (partially) specified

$$\dot{x} = f(x,u), x(0) = x_0$$

*This work is being done by various members of the class of 2012 and we wish to optimize

$$J = \psi(x(t)) + \int_0^T l(x(t), u(t))dt$$

with specific terminal values $X_1(T), ..., X_r(T) \quad (r \leq n)$

The approach for the solution again involves infinite dimensional Lagrange Multipliers

$$\hat{J} = \psi(x,t) + \int_0^T H(\lambda(t), x(t), u(t)) - \lambda(t)^T \dot{x}(t)dt$$

with

$$H(\lambda, x, u) = \lambda^T f(x,u) + l(x,u).$$

suppose

$$x(T) = \begin{pmatrix} x_1(T) \\ . \\ . \\ . \\ x_r(T) \\ x_n(T) \end{pmatrix}$$

with $X_1(T), ..., X_r(T)$ specified.

In order to first orderize J

$$\delta\bar{J} = \frac{\delta\psi}{\delta x}(x^*(T) - \lambda(T))^T \delta x(t) + \int_0^T \frac{\delta H}{\delta x}(\lambda, x^*, u^*) + \dot{\lambda}(t))^T \delta x(t)dt + \int_0^T \frac{\delta H}{\delta x}(\lambda, x^*, u^*)\delta u(t)dt$$

As in the last lecture we choose $\lambda(t)$ to satisfy $\lambda(\dot{t}) = \frac{-\delta H}{\delta x}$ making the 2nd term in the above equation vanish since $\delta_{xj}(T) = 0 for i = 1, ..., r$ The corresponding costate component $\lambda_j(T)$ are not constrained so now the P.M.P. becomes

$$\dot{x} = f(x,u), x(0) = x_0$$

$$\dot{\lambda} = \frac{-\delta f}{\delta x}^T \lambda(t) - \frac{\delta l}{\delta x}$$

$$\lambda_{r+1}(T) = \frac{\delta\psi}{\delta x}(x(T))_{r+1}$$

....

$$\lambda_n(T) = \frac{\delta\psi}{\delta x}(x(T))_n$$

$$\frac{\delta H}{\delta U} = 0,$$

$$X_1(T) = x_1$$

$$X_2(T) = X_2$$

$$\downarrow$$

$$X_r(T) = X_r$$

Example Finite Dimensional LTI system

We would like to steer

$$\dot{x} = Ax + Bu$$

from $x(0) = x_0$ to $x(T) = x_1$ so as to minimize

$$J = \frac{1}{2}\int_0^T ||u(t)||^2 dt$$

To solve, first let us write down the Hamiltonia

$$H(\lambda, x, u) = \lambda^T Ax + \lambda^T Bu + \frac{1}{2}||u||^2$$

the P.M.P. necessary conditions say that there is a *costate* $\lambda(t)$, which together with the optimizing values of "x" and "u" satisfy

$$\dot{\lambda} = -A^T \lambda(t)$$

$$B^T \lambda(t) + u(t) = 0$$

$$\dot{\lambda} = Ax + Bu$$

From the first equation $\lambda(t) = e^{-A^T t}\lambda_0$ for some $\lambda_0 \in \mathbb{R}^n$

$$u(t) = -B^T \lambda(t)$$

$$= -B^T e^{-A^T} \lambda_0$$

Now λ_0 must be chosen s.t. the boundary conditions $x(0) = x_0$, $x(t) = x$, are met, so from the variation of constants formula

$$x(t) = e^{at} x_0 - \int_0^T e^{A(t-s)} BB^T e^{-A^T s} ds \lambda_0$$

having the boundary conditions met requires that

$$x(t) = e^{at}(x_0 - \int_0^T e^{-A(s)} BB^T e^{-A^T s} ds\lambda_0$$

In other words

$$\int_0^T e^{-A(s)} BB^T e^{-A^T s} ds\lambda_0 = (x_0 - e^{-AT} x_1)$$

Hence,

$$\lambda_0 = W^{-1}(x_0 - eATx_1)$$

where

$$\int_0^T e^{-A(s)} BB^T e^{-A^T s} ds$$

Note: That W is an invertible n x n matrix if the system is controllable.

Summary: The control law that steers $\dot{x} = Ax + Bu from x(0) = x_0 to X(T) = x_1$ so as to minimize the norm of control squared is integrated

$$\int_0^T ||u(t)||^2 dt$$

Is given by

$$u(t) = -B^T e^{(-A^T t)} \lambda_0$$

where λ_0 is the soltuion to $x_0 - e^{-A^T} x_1 = W\lambda_0$ with

$$\int_0^T e^{-A(s)} BB^T e^{-A^T s} ds$$

Other linear quadratic Optimization problems:

Free Endpoint Problem (not specifying the final endpoint)

$$\dot{x} = Ax + Bu$$

steer this system so as to minimize

$$J = \frac{1}{2} \int_0^T x^T(t) Q x(t) + u^T(t) R u(t) dt$$

Where R is symmetric and positive definite, and Q is symmetric and positive semi-definite.

For this problem the Hamiltonian is $\lambda^T Ax + \lambda^T Bu + \frac{1}{2}u^T Ru$ the P.M.P necessary conditions are

$$x = A^T\lambda - Qx; \lambda(T) = 0$$

$$\dot{x} = -A^T\lambda - Qx$$

$$Ru + B^T\lambda = 0$$

R is invertable because it is positive definite $u(t) = -R^{-1}B^T\lambda(t)$

$$\dot{x} = Ax - BR^{-1}B^T\lambda$$

$$\dot{\lambda} = -A^T\lambda - Qx$$

$$x(0) = x_0, \lambda(T) = 0 \leftarrow fromP.M.P$$

suppose we can write

$$\lambda(T) = M(t)x(t)$$

$$\dot{\lambda} = M\dot{M}x + M\dot{x}$$

from x equation

$$= \dot{M}x + MA - MBR^{-1}B^T Mx$$

from the λ equation

$$= -A^T Mx - Qx$$

This will be satisfied for any X(t) of M(t) satisfies the matrix Riccoti equation

$$\dot{M} + A^T M + MA - MBR^{-1}B^T M + Q = 0(R)$$

To meet the boundary conditions $\lambda(T) = 0$ we require that M(t)=0

The question of Existance of M

Consider the Scalar Riccoti equation

$$\dot{M} = -2M + M^2 + 1$$

$$M(T) = 0$$

(This would be associated with Optimization)

$$\dot{x} = x + u$$

$$J = \int_0^T u^2(t) - x^2(t) dt$$

note: $\dot{m} = (m-1)^2$ and get

$$\dot{k} = k^2$$

This is a solution of the form

$$k(t) = \frac{\alpha}{1 - \alpha t}$$

$$\dot{k}(t) = \frac{\alpha^2}{(1 - \alpha t)^2} = k(t)^2$$

One can prove that the solution is unique. Choose α s.t. M(t) = 0 i.e. K(T) = -1 then

$$\frac{\alpha}{1 - \alpha t} = -1 \rightarrow \alpha = \frac{1}{T - 1}$$

For example when T=2 and $\alpha = 1$ and $k(t) = \frac{1}{1-T}$

$$M(t) = \frac{1}{1 - t}$$

Which blows up at t = 1

So the quadratic is not always the best option since it has a finite blow up times

We state the following without proof

Theorem: If A,B is a controllable pair, $Q = Q^T$ is positive semi-definite, then the solution to the Riccoti equation (T) passing through 0 at t=T exists and is positive semi-definite on an interval $[t_0, T]$ for any $t_0 < T$

<u>Previw of the next time</u>

<u>Theorem:</u> Suppose A,B is a controllabe pair. The conrol law that steers

$$\dot{x} = Ax + Bu$$

so as to minimize

$$\int_0^T x^T Q x + u^T R u dt$$

is given by

$$u(t) = -R^{-1} B^T M(t) x(t)$$

where M satisfies

$$\dot{M} + MA + A^T M - MBR^{-1} B^T M + Q = 0$$

$$M(T) = 0$$