Background on Linear Algebra - Lecture 2 *

September 6, 2012

1 Introduction

Recall from your math classes the notion of vector spaces and fields of scalars. We shall be interested in finite dimensional vector spaces, and the scalar fields of interest will be real $\mathbb{R}$ and $\mathbb{C}$ complex numbers.

Because the vector spaces of interest are finite dimensional, there is no loss of generality in thinking of them as $\mathbb{R}^n$ and $\mathbb{C}^n$ for appropriate position integer n.

consider the linear transformation

$L : \mathbb{C}^n \to \mathbb{C}^n$

Recall that in a linear transformation is a function such that for all vectors x, y ϵ $\mathbb{C}^n$ and scalars α and $\beta \in \mathbb{C}$

L(αx + βy) = (αLx + βLy)

If we choose a basis $\{v_i, ..., v_n\}$ for $\mathbb{C}^n$

then L may be represented by a matrix as follows

$$Lv_j = \sum_{i=1}^{n} \alpha_{ij} v_i$$

$$L \sim (\alpha_{ij}) = A)$$

where matrix A has its ij-th entry is α_{ij}

*This work is being done by various members of the class of 2012

What if we choose another basis

$\{w_1, ..., w_n\}$ and write

$$\text{L} \sim (b_{ij}) = B \ ?$$

While it might look like a given linear transformation can be represented by any of a given number of arbitrary different matrices, this is not a case

Note: any two basis are related by a "change of basis" matrix

$$w_j = \sum_{i=1}^{n} p_{ij} v_i$$

where the p_{ij}'s are uniquely determined scalars. Write $P \sim (p_{ij})$

Note that P is an invertable matrix. This follows easily because the basis $(v_i \ldots, v_n)$ may be expressed in terms of the basis $(w_i, \ldots, w_n)$

$$v_j = \sum_{i=1}^{n} q_{ij} w_i$$

$$(q_{iy}) \sim Q$$

clearly $Q = P^{-1}$ (Proof for this is left to the students.)

On the one hand

$$Lw_j = \sum_{\ell=1}^{n} p_{\ell j} L v_\ell$$

$$= \sum_{\ell=1}^{n} (\sum_{k=1}^{n} \alpha_{k\ell} v_k)$$

$$= \sum_{k=1}^{n} (\sum_{\ell=1}^{n} \alpha_{k\ell} p_{\ell j}) v_k.$$

On the other hand

$$
\begin{aligned}
Lw_j &= \sum_{i=1}^{n} \beta_{ij} w_i \\
&= \sum_{i=1}^{n} \beta_{ij} (\sum_{k=1}^{n} p_{ki} v_k) \\
&= \sum_{k=1}^{n} (\sum_{i=1}^{n} p_{ki} \beta_{ij}) v_k
\end{aligned}
$$

$$\sum_{k=1}^{n} (\sum_{i=1}^{n} p_{ki} \beta_{ij}) v_k = \sum_{k=1}^{n} (\sum_{\ell=1}^{n} \alpha_{k\ell} p_{\ell j}) v_k.$$

Because the $v_k's$ form a basis , the coefficients of the $v_k's$ in this equation are equal term by term

$$\sum_{i=1}^{n} p_{ki} \beta_{ij} = \sum_{l=1}^{n} \alpha_{kl} p_{\ell j} \quad (k = 1, ...n),\ (j = 1, ..., n)$$

The left hand side is the kj entry in the matrix product AP. The right hand side is the kj entry in the matrix product PB.

Theorem

Two square matrices A,B represent the same linear transformation if and only if there is a non singular matrix P such that

$B = P^{-1} A P$

Proof

The calculation shows that there is a matrix P such that

$PB = AP$

Since P is invertible, the conclusion of the theorem follows.

Terminology

$A \rightarrow P^{-1} A P$ is called a *similarity transformation.*

Two matrices represent the same linear transformation if and only if they are similar

<u>Goal</u>

When we look for matrix representations the goal will be to find expressions that are as simple as possible—

for instance, we would like matrices to be sparse i.e. to have as many zero elements as possible

<u>Example</u>

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Propose we let

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

then it is (almost) obvious that

$$P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

we find that

$$\begin{aligned} B &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}. \end{aligned}$$

Generally speaking square matrices with all non-zero elements (if any) confined to the principal diagno are as sparse as possible. under the operation of similarity transformations.

<u>Theorem</u>

Any square $n \times n$ matrix with n distinct eigenvalues can be put into diagonal form by a change of basis.

<u>Proof</u>

Let $\{e_1, ..., e_n\}$ be any set of eigenvectors corresponding to the distinct eigenvalues $(\lambda_1, ..., \lambda_n)$

$Ae_j = \lambda_j e_j \ (j = 1, ..., n)$

Think of the eigenvectors as n-tuples of scalars, and if we think of these n-tuples as column vectors, we can put them side by side to form an $n \times n$ matrix

$M = (e_1 \cdots e_n)$

$AM = (\lambda_i e_i :: \lambda_n e_n) = M\Lambda$
where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

This proves the theorem.

Exercise: M is non-singular <u>hint</u>: $(e_1, ..., e_n)$ form a basis

<u>Example</u>

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$$

this has an eigenvalue that are distinct if and only if a $\neg 1$

might as well take the eigen vector corresponding to 1 to be

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note: eigenvectors remain eigenvectors if they are multiplied by a nonzero scalar

we can take the eigen vector corresponding to a to be

$$\vec{e}_a = \begin{pmatrix} 1 \\ a-1 \end{pmatrix}$$

A diagnolizing transformation M such that

$$M^{-1} \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} M$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

is

$$M = \begin{pmatrix} 1 & 1 \\ 0 & a-1 \end{pmatrix}.$$

Note that as a $\to 1, \vec{e}_a \to e_1$, and moreover, M becomes singular. What to do?

Homework - go back and review what you know about elementary linear algebra

next: where to go when eigenvalues are not distinct

2 Recommended reading

B.Noble & J.W. Daniel, Applied Linear Algebra, Prentice Hall, 1977

Gilbert Strang, Introduction to Linear Algebra 4th Edition, Wellesley Cambridge Press

Steven Roman, Advanced linear Algebra (Graduate texts in mathematics), Springer 3rd Edition

F.R. Gantmacher, Matrix theory, Chealsea, NY 1960