Lecture 19 Observability Canonical Form and the Theory of Observers

November 15, 2012

Observability Canonical Form

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

Suppose this is observable:

Let \( S = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \) Assume scalar output.

Let \( z = Sx \). Then

\[
\dot{z} = SAS^{-1}z + SBu \\
y = CS^{-1}z
\]

Let \( S^{-1} = \begin{pmatrix} S_1 & \cdots & S_n \end{pmatrix} \)

\[ CA^kS_j = \begin{cases} 1 & k = j - 1 \\ -a_{j-1} & k = n \\ 0 & \text{otherwise} \end{cases} \]

Hence,

\[
SAS^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}
\]

\[
CS^{-1} = \begin{pmatrix} 1, & 0, & 0, & \cdots, & 0 \end{pmatrix}
\]

*This work is being done by various members of the class of 2012*
Summary: For Single Input/Single Output systems

\[
\dot{x} = Ax + bu \\
y = Cx
\]

The system is said to be in controllability canonical form if:

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{pmatrix},
\quad b = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
\]

It is in observability canonical form if

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{pmatrix},
\quad c = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Observers

Suppose we are given

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

and we wish to estimate \(x_0\). "After a while", the state is given by

\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma
\]

If \(A\) is a stable matrix (i.e. eigenvalues in the l.h.p), then the dependence of \(x(t)\) on \(x_0\) diminishes over time. Thus, eventually depends only on the second term. If we run another system

\[
\dot{z} = Az + Bu; \quad z(0) = z_0
\]

in parallel, then the over time \(e = z - x\) decreases to 0.
Proof. $e = z - x$ satisfies the homogeneous linear ordinary differential equation $\dot{e} = Ae$.

Identity Observer

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}$$

We propose an observer of the form

$$\begin{align*}
\dot{z} &= Az + E(y - Cz) + Bu, \quad z(0) = z_0
\end{align*}$$

where $E$ is a coefficient matrix to be specified.

**Theorem.** If the system (1) is observable, the coefficients of the characteristic polynomial of $A - EC$ may be selected arbitrarily by appropriate choice of $E$.

Proof. The result follows from the eigenvalue placement problem follow the controllable pair $(A^T, CT)$. Hence, we can find a $K$, such that $A^T + CTK$ has any chosen characteristic polynomial. Let $E = -K^T$. Then $A - EC$ has the same characteristic polynomial.

Note that if $e = z - x$

$$\begin{align*}
\dot{e} &= \dot{z} - \dot{x} \\
&= Az + E(y - Cz) + Bu - Ax - Bu \\
&= Az + E(Cx - Cz) - Ax \\
&= (A - EC)e
\end{align*}$$

If the eigenvalue of $A - EC$ are in the l.h.p, then $e(t) \to 0$ asymptotically.

**Reduced Order Observers**

We don’t need to propagate an n-dimensional reconstruction of the state since we directly observe p-dimensions in the form of the output $y$.

Thus, we seek to reconstruct $n - p$ observer states. We are given the system

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}$$

We assume that the $p \times n$ matrix $C$ has full rank $p$. Let $v$ be any $(n - p) \times n$ matrix such that $p = \begin{pmatrix} v \\ c \end{pmatrix}$ is non-singular.

The set of independent vectors that are the row of $C$ can be completed by an additional $n - p$ independent row vectors to yield a basis for $\mathbb{R}^n$. These rows are grouped together to gain the $(n - p) \times n V$, making $p = \begin{pmatrix} v \\ c \end{pmatrix}$ invertible.
Let $\bar{x} = P\bar{z}$, and write $\bar{x} = \begin{pmatrix} w \\ y \end{pmatrix}$. In terms of $\bar{x}$, the system dynamics are of the form:

$$
\begin{pmatrix}
\dot{w} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u
$$

It is possible to extract from this system a system of order $n - p$ which has as inputs the known quantities $u, y$.

Let $E$ be $(n - p) \times p$ matrix, and subtract $E \cdot (\text{bottom part of the system})$ from the top.

$$
\dot{w} - E\dot{y} = (A_{11} - EA_{21})w + (A_{12} - EA_{22})y + (B_1 - EB_2)u \\
= (A_{11} - EA_{21})(w - Ey) + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u
$$

Letting $v = w - Ey$, this reduced order system takes the form:

$$
\dot{v} = (A_{11} - EA_{21})v + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u
$$

where $y, u$ are known inputs. Form an observer by copying this system

$$
\dot{z} = (A_{11} - EA_{21})z + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u
$$

The error in the $z$ estimate of $\dot{z}$ propagates according to

$$
\dot{e} = \dot{z} - \dot{v} \\
= (A_{11} - EA_{21})e
$$

the rate at which $e \to 0$ depends on the eigenvalues of $(A_{11} - EA_{21})$. This can be placed arbitrarily provided that $(A_{11}, \ A_{21})$ is an observable pair.

**FACT.** *If the original system $(A, B, C)$ is observable, then $(A_{11}, \ A_{21})$ must be an observable pair.*

**Proof.** Left to students as homework. □

Reconstructing the original state: Since $v = w - Ey$, $z \sim v$, $\hat{w} = z + Ey$ is my observer estimate of $w$. The state estimate is thus:

$$
\hat{x} = P^{-1} \begin{pmatrix} \hat{w} \\ y \end{pmatrix} = P^{-1} \begin{pmatrix} \hat{z} + Ey \\ y \end{pmatrix}
$$

**Observers in Feedback Loops (the Separation Principle)**

$$
\dot{x} = Ax + Bu \\
y = Cx \\
\dot{z} = Az + E(y - Cz) + Bu
$$
We would like to choose a gain matrix $K$, defining a feedback $u = Kx$ such that the resulting closed loop system is stable. Unfortunately, $x$ is not available. For this purpose, assuming the observer works, we can by feeding back $u = Kz$, the resulting closed loop system is of the form:

$$\dot{x} = Ax + BKz$$

$$\dot{z} = (A - EC)z + ECx + BKz$$

Let $e = z - x$, then the above may be rewritten as:

$$\dot{x} = (A + BK)x + BKe$$

$$\dot{e} = (A - EC)e$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BK & BK \\ 0 & A - EC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

**FACT.** $\det \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} = \det(X)\det(Z)$, where $X$ and $Z$ are square blocks.

**Proof.**

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix}$$

$$\det \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} = 1 \det \begin{pmatrix} I_{n-1} & 0 \\ 0 & Z \end{pmatrix} = \cdots = \det(Z)$$

Similarly,

$$\det \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix} = \det(X)$$

We know that

$$\det \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix} = \det \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \det \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix}$$

and the result (fact) follows. From this, it is easy to see that the eigenvalues of $\begin{pmatrix} A + BK & BK \\ 0 & A - EC \end{pmatrix}$ are the eigenvalues of $A + BK$ together with the eigenvalues of $A - EC$.  \qed