Dynamic Systems Theory-State-space Control - Lecture 16 *

November 1, 2012

1 Summary of Controllability and Observability

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$y = c(t)x(t)$$
$$x \sim dim\ n, y \sim dim\ m, u \sim dim\ p$$

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B(t)^T\Phi(t_0, t)^T dt$$
$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0) dt$$

These grammians tell the whole story.
The system is controllable $\Leftrightarrow$ $W(t_0, t_1)$ is invertible.
The system is observable $\Leftrightarrow$ $M(t_0, t_1)$ is invertible.
For the constant coefficient case, the relevant objects are
$(B, AB, \dots, A^{n-1}B)$ controllability: rank n
$(C, CA, \dots, CA^{n-1})^T$ observability: rank n

2 Discrete time case

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k)$$

*This work is being done by various members of the class of 2012

$$
\begin{aligned}
x(N+1) &= A(N)x(N) + B(N)u(N) \\
&= \underbrace{A(N)\ldots A(1)}_{\Phi(N,1)} x(1) + A(N)\ldots A(2)B(1)u(1) + \ldots + B(u)u(N) \\
&= \Phi(N,1)x(1) + \sum_{j=1}^{N-1} \Phi(N,j+1)B(j)u(j) + B(N)u(N)
\end{aligned}
$$

2.1 Discrete Time Theorem

<u>Discrete Time Theorem</u>: There exists a control sequence $u(1), \ldots, u(N)$ that steers the state of

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

from $x(t) = x_0$ to $x(N) = x_1$ if and only if

$$x_1 - \Phi(N-1,1)x_0$$

belongs to the range space of

$$W = \sum_{j=1}^{N-2} \Phi(N-1,j+1)B(j)B(j)^T\Phi(N-1,j+1)^T + B(N-1)B(N-1)^T$$

<u>Proof</u>: Suppose there is an η such that

$$x_1 - \Phi(N-1,1)x_0 = W\eta$$

Let

$$u(j) = \begin{cases} B(j)^T\Phi(N-1,j+1)^T\eta & j = 1, \ldots N-2 \\ B(N-1)^T\eta & \text{for } j = N-1 \end{cases}$$

Then

$$
\begin{aligned}
x(N) = \Phi(N-1,1)x_0 &+ \sum_{j=1}^{N-2} \Phi(N-1,j+1)B(j)(B(j)^T\Phi(N-1,j+1)^T)\eta \\
&+ B(N-1)B(N-1)^T\eta \\
&= \Phi(N-1,1)x_0 + W\eta \\
&= x_1
\end{aligned}
$$

Going the other way, suppose that $x_1 - \Phi(N-1,1)x_0$ is not in the range space of W. Then there exists a $y \in \mathbb{R}^n$ such that $y \circ (x_1 - \Phi(N-1,1)x_0) \neq 0$, but $W^Ty = 0$. Following the same reasoning as in the continuous time case, this would imply

$$0 = y^T W y = \sum_{j=1}^{N} ||B(j)^T \Phi(N,j)^T y||^2$$
$$\Rightarrow B(j)^T \Phi(N,j)^T y = 0 \quad \text{for } j = 1, \ldots N$$

This contradiction shows that if there is an input sequence achieving specified boundary conditions for all possible choices of boundary conditions, then W has full rank.

2.2 Constant coefficient case

$$\begin{aligned} x(N+1) &= Ax(N) + Bu(N) \\ &= A^2 x(N-1) + ABu(N-1) + Bu(N) \\ &\ldots \\ &= A^N x(1) + A^{N-1}Bu(1) + A^{N-2}Bu(2) + \ldots + ABu(N-1) + Bu(N) \end{aligned}$$

The controllability rank condition [rank $(B, AB, \ldots, A^{N-1}B) = n$] is equivalent to the condition of whether or not we can find a sequence $u(1), \ldots, u(N)$ which steers between x_0 and x_1 for any choice of values $x_0, x_1 \in \mathbb{R}^n$.

We can always choose $N \leq n$. Hence the controllability condition is

$$B, AB, \ldots, A^{n-1}B$$

has rank n.

2.3 Observability in the constant coefficient discrete time case

$$y(k) = Cx(k) \qquad x(k+1) = Ax(k)$$

we observe

$$\left.\begin{array}{ccc} y(k) & = & Cx(k) \\ y(k+1) & = & CAx(k) \\ & \vdots & \\ y(n-1+k) & = & CA^{n-1}x(k) \end{array}\right\} \text{observe eq.}$$

We can always solve the observe eq. for $x(k)$ in terms the sequence $y(k), y(k+1), \cdots, y(k+n-1) \Leftrightarrow$ rank $(C, CA, \cdots, CA^{n-1})^T = n$

3 Sampled systems hybrid continuous discrete systems

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{1}$$

By samlpling the state every h seconds, we get

$$y(kh) = C(kh)x(kh)$$

Letting

$$\overline{x}(k) = x(kh), \overline{A}(k) = \Phi((k+1)h, kh)$$

$$\overline{B}(k) = \int_{kh}^{(k+1)h} \Phi((k+1)h, s)B(s)ds$$

$$\overline{C}(k) = c(kh)$$

We find the state and output dynamics of

$$x(k+1) = \overline{A}(k)\overline{x}(k) + \overline{B}(k)\overline{u}(k) \tag{2}$$

is the same as a sampled system, sampled every h units of time

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u_c(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{3}$$

where

$$u_c(t) = \overline{u}(k) \qquad \text{for} \quad kh \leq t < (k+1)h$$

(2) is called the sampled version of (1).

Note: Constant coefficient continuous time systems give rise to constant coefficient sampled systems.

$$\overline{A} = e^{Ah}$$

$$\overline{B} = \int_{kh}^{(k+1)h} e^{A[(k+1)h-s]}Bds = \int_{0}^{h} e^{A(h-s)}Bds$$

The Laplace transform of

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u_c(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

is

$$\hat{y}(s) = C(Is - A)^{-1}B\hat{u_c}(s)$$

A discrete sequence $x(0), x(1), x(2), \cdots$ has an associated z-transform

$$\hat{x}(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

Apply the z-transform to both sides of

$$\begin{aligned} x((k+1)h) &= Ax(kh) + Bu(kh) \\ y(kh) &= Cx(kh) \end{aligned}$$

$$\sum_{k=0}^{\infty} x((k+1)h)z^{-k}) = A \sum_{k=0}^{\infty} x(kh) z^{-k} + B \sum_{k=0}^{\infty} u(kh) z^{-k}$$

The left hand side can be written

$$z[\sum_{k=0}^{\infty} x(kh) z^{-k} - x(0)]$$

Assume that $x(0) = 0$
Denote

$$Y(z) = \sum_{k=0}^{\infty} y(kh) z^{-k}$$

$$X(z) = \sum_{k=0}^{\infty} x(kh) z^{-k}$$

$$U(z) = \sum_{k=0}^{\infty} u(kh) z^{-k}$$

$$\begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) \\ &= C(Iz - A)^{-1} BU(z) \end{aligned}$$

We wish to have a formal procedure for relating the z-transform of a sample system to the Laplace transform of the continuous time system from which it was obtainted.

Given a continuous signal $f(t)$, we define the sampled represatation to be

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(kh) \delta(t - kh)$$

where $\delta(t)$ is the Direc delta function defined formally by

$$\int_{-\infty}^{\infty} y(s) \delta(s) ds = g(0)$$

The Laplace transform of the sampled system $f^*(t)$ is

$$\begin{aligned}\sum_0^{\infty} f^*(t)e^{-st}dt &= \int_0^{\infty} \sum_{k=-\infty}^{\infty} f(kh)e^{-st}\delta(t-kh)dt \\ &= f(0) + f(h)e^{-sh} + f(2h)e^{-s2h} + \cdots \\ &= \sum_{k=0}^{\infty} f(kh)(e^{-sh})^k\end{aligned}$$

This is the z-transform provided $f(t) = 0$ for $t < 0$ and provided we make the identification $z = e^{sh}$.

REMARKS ON THE ZERO-ORDER HOLD:

Given the sampled signal $\{f(kh), k = 0, 1, 2 \cdots\}$, one useful reconstruction is

$$f_c(t) = f(kh) \qquad \text{for} \quad kh \leqslant t < (k+1)h$$

Applying this to our sampled control input $u_c(t)$, the Laplace transform is

$$\begin{aligned}\hat{u}_c(s) &= \int_0^{\infty} e^{-st}u_c(t)dt \\ &= \sum_{k=0}^{\infty} \int_{kh}^{(k+1)h} e^{-st}u(kh)dt \\ &= \sum_{k=0}^{\infty} u(kh)(-\frac{1}{s}e^{-st} \mid_{t=kh}^{t=(k+1)h}) \\ &= \sum_{k=0}^{\infty} u(kh)\frac{1}{s}(e^{-skh} - e^{-s(k+1)h}) \\ &= \sum_{k=0}^{\infty} u(kh)e^{-skh}\frac{1}{s}(1 - e^{-sh}) \\ &= \frac{1}{s}(1 - e^{-sh})\mathfrak{L}(u^*(t)) \\ &= \frac{1}{s}(1 - e^{-sh})u_c(z)\end{aligned}$$

Note that $\frac{1}{s}(1 - e^{-sh})$ is the Laplace transform of the <u>zero-order hold</u>; i.e. the Laplace transform of

$$x(t) = \begin{cases} 1 & 0 \leq t < h \\ 0 & \text{elsewhere} \end{cases}$$