# Dynamic Systems - State Space Control - Lecture 14 *

October 23, 2012

## 1 Equilibirum Points

**THEOREM:** A necessary a sufficient condition for the equilibrium point of $x_dot = Ax + b$ is that the Eigenvalues of A have negative real parts. That is to say that the eigenvalues lie in the left half complex planes.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State feedback $u(t) = Kx(t)$
Compared to output feedback $u(t) = Ky(t)$

The closed loop system with state feedback is:

$$\dot{x} = (A + Bk)x$$

Examples:

1. RLC Circuit

   Insert drawing here

   Kirchoff's Law (Voltage Drops)

$$V = IR$$
$$V = \mathrm{d}FL/\mathrm{d}t$$
$$V = 1/C \int I \mathrm{d}s$$

*This work is being done by various members of the class of 2012

$$L = \mathrm{d}^2 I/\mathrm{d}t^2 + R\mathrm{d}I/\mathrm{d}t + 1/cI = E(t)$$

RLC circuit is controlled by voltage flux

2. Positioning a Mass Spring Dashpot System

Insert drawing here

$$m\ddot{x} + c\dot{x} + kx = u(t)(c > 0)$$

First orderizing the system yields

$$x_1 = x, x_2 = \dot{x}$$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial is

$$\lambda^2 + c\lambda/m + k/m = 0$$

$$\text{Roots } \lambda_i = -c/2m \pm 1/2m\sqrt{c^2 - 4km}$$
$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

If there is a large damping $c^2 >> 4km$
Insert drawing here

If either k or m or both are large

$$x(t) = a_1 e^{-c/2m*t} cos(1/2m\sqrt{|c^2 - 4km|}t) + a_2 e^{-c/2m} sin(1/2m\sqrt{|c^2 - 4km|}t)$$

What can be achieved with feedback

$$u(t) = -k_p x_1(t) - k_v x_2(t)$$
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -k_p x_1(t) - k_v x_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -k/m - k_p & -c/m - k_v \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The closed loop characteristic polynomial equation is

$$\lambda^2 + (c/m + k_v)\lambda + k/m + k_p = 0$$

Second order control systems

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0, \zeta > 0$$

$\zeta$ is the damping ratio
$\omega$ is the natural frequency

The three cases of interest are $\zeta < 1, \zeta = 1, \zeta > 1$

Why these are of interest

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

$$s = -\zeta\omega \pm \omega\sqrt{\zeta^2 - 1}$$

Case $\zeta < 1$

Let $a = -\zeta\omega, \omega > 0$
Then $a^2 = \zeta^2\omega^2 < \omega^2$
Hence we can choose b s.t $a^2 + b^2 = \omega^2$
The differential equation becomes:

$$\ddot{x} - 2a\dot{x} + (a^2 + b^2)x = 0$$

First orderize as follows

$$x_1 = x$$

$$x_2 = \dot{x} - ax/b$$

Then

$$\dot{x_1} = ax_1 + bx_2$$

$$\dot{x_2} = -bx_1 + ax_2$$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We have computed

$$\Phi(t, 0) = e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t}$$

$$= \begin{pmatrix} e^{at}cos(bt) & e^{at}sin(bt) \\ -e^{at}sin(bt) & e^{at}cos(bt) \end{pmatrix}$$

$$x(t) = x_1(t) = e^{at}cos(bt)x_1(0) + e^{at}sin(bt)x_2(0)$$

Insert drawing here

$$= Ce^(at)sin(bt + \phi)$$
$$C = \sqrt{x_1(0)^2 + x_2(0)^2}$$
$$\Phi = arctan(x_1(0), x_2(0)$$

If a $> 0$

Insert drawing here

Case $\zeta > 1$

$$\lambda_1 = -\zeta\ omega - \omega\sqrt{\zeta^2 - 1}$$
$$\lambda_2 = -\zeta\ omega + \omega\sqrt{\zeta^2 - 1}$$

These roots are either both positive or both negative. Again let $a = -\zeta\omega, \omega > 0$
Then $a^2 = \zeta^2\omega^2 > \omega^2$ Hence there is a $b^2 < a^2$sich that $a^2 - b^2 = \omega^2$
First orderize as the system

$$\dot{x_1} = ax_1 + bx_2$$

$$\dot{x_2} = bx_1 + ax_2$$
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The transition matrix is

$$e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} t} e^{\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} t}$$
$$e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} t} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix}$$
$$e^{\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + 1/2 \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \\ +... & \end{pmatrix}$$

The even terms sum to

$$\begin{pmatrix} e^{bt} + e^{-bt}/2 & 0 \\ 0 & e^{bt} + e^{-bt}/2 \end{pmatrix}$$

The odd terms sum to

$$\begin{pmatrix} 0 & e^{bt} - e^{-bt}/2 \\ e^{bt} - e^{-bt}/2 & 0 \end{pmatrix}$$

The transition matrix is

$$\begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix} \begin{pmatrix} e^{bt} + e^{-bt}/2 & e^{bt} - e^{-bt}/2 \\ e^{bt} - e^{-bt}/2 & e^{bt} + e^{-bt}/2 \end{pmatrix}$$

Assuming (without loss of generality) $b > 0$ we have $|a| > b$

$$x(t) = C_1 e^{(a+b)t} + C_2 e^{(a-b)t}$$

$a < 0$
Insert drawing here

$a > 0$
Insert drawing here

Case $\zeta = 1$

$$\dot{x_1} = x$$

$$\dot{x_2} = \dot{x_1} + \omega x_1$$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -\omega & 1 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x(t) = e^{-\omega t}(x_1 cost + x_2(0))$$

Summary -

Pole locations and closed loop dynamics. Poles occur as complex conjugate pairs.

Insert drawing here

# 2 Mass Spring System Reprise

$$m\ddot{x} + c\dot{x} + km = u(t)$$

Assuming we can measure $x, \dot{x}$ and feed them back

$$u(t) = -k_p x(t) - k_v \dot{x}(t)$$

The closed loop system is

$$m\ddot{x} + (c + kv)\dot{x} + (k - k_p)x = 0$$

$$k + k_p/m = \omega^2$$

$$c + k_v/m = 2\zeta\omega$$

$$\zeta = c + k_v/2\sqrt{k + k_p}\sqrt{m}$$

When $\zeta < 1$

$$\omega^2 = a^2 + b^2$$

$$\zeta = -a/\omega = -a/\sqrt{a^2 + b^2}$$

When $\zeta > 1$

$$\omega^2 = a^2 - b^2$$

$$\zeta = -a/\omega = -a/\sqrt{a^2 - b^2}$$

When $\zeta = 1$

$$\omega^2 = a^2 + b^2$$

$$x(t) = C_1 e^{(a+b)t} + C_2 e^{(a-b)t}, a < 0, b > 0, |a| > b$$

If —a— is very large compared to b $\zeta = 1$

$$x(t) = Ce^{at}$$

Insert drawing here

If kv is very large $\zeta >> 1$ and —a—, b have similar magnitude

$$a + b = -\omega < 0$$

$$x(t) = C_1 e^{-\omega t} + C_2 e^{(-2b-\omega)t}$$

Insert drawing here

When $\zeta = 1$ there is fast damping but there is danger if there are modeling errors of the system actually being underdeveloped

# 3 Roth, Hurwitz Asymptotic Stability

`-> G(s) ->`
Suppose g(s) is a proper rational transfer function, $g(s) = n(s)/d(s)$
Asymptotic stability depends on the zeros of g(s)

$$d(s) = s^n + a_n - 1s^{n-1} + ... + a_0$$

They need to be in the left half plane.
d(s) is said to be a Hurwitz polynomial when they are.
Since d(s) is the product of factors of the form s+a and $s^2 + b_1 s + b_0$
With $a, b_1, b_0$ real and positive, a necessary condition for d(s) to be Hurwitz is that all coefficients $a_k$ be positive. This however is not sufficient.

Exercise 1. For a polynomial having all coefficients positive such that it is not Hurwitz.

The Routh table associated with $d(s) = s^n + a_n - 1s^{n-1} + ... + a_0$

$$\begin{array}{c|c|c|c|c} a_0 & a_2 & a_4 & a_6 & ... \\ a_1 & a_3 & a_5 & a_7 & ... \\ \hline b_1 & b_2 & b_3 & b_4 & ... \\ c_1 & c_2 & c_3 & c_4 & ... \\ d_1 & d_2 & d_3 & d_4 & ... \end{array}$$

The entries below the two rows of a's are"

$$b_1 = a_1 a_2 - a_0 a_3 / a_1$$

$$b_2 = a_1 a_4 - a_0 a_5 / a_1$$

$$b_3 = a_1 a_6 - a_0 a_7 / a_1$$

$$c_1 = b_1 a_3 - b_2 a_1 / b_1$$

$$c_2 = b_1 a_5 - b_3 a_1 / b_1$$

$$d_1 = c_1 b_2 - b_1 c_2 / c_1$$

etc ...

**<u>ROUTH-HURWITZ THEOREM:</u>** The number of sign changes in the left-hand column as you go down is equal to the number of zeros in the right half plane.

Example : $d(s) = s^4 + s^3 + s^2 + 11s + 10$

Routh Table :

$$\begin{array}{c|c|c} 10 & 1 & 1 \\ 11 & 1 & 0 \\ \hline 1/11 & 1 & 0 \\ -120 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

$$b_1 = 11 * 1 - 1 * 10/11$$

$$b_2 = 1 * 1 - 1 * 0/11$$

$$c_1 = 1/11 - 1 * 11/1/11 = -120$$

Two sign changes in the first column means two zeros in the right half plane.

Reference : Lerrant, Lepschy, Viavo. 1999. "A Simple Proof of the Routh Test". IEEE Transactions on Automatic Conrol. 44(6). 1306-1309.