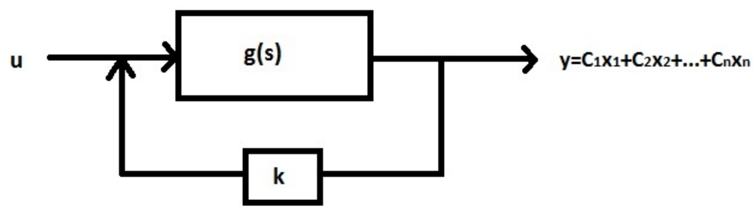
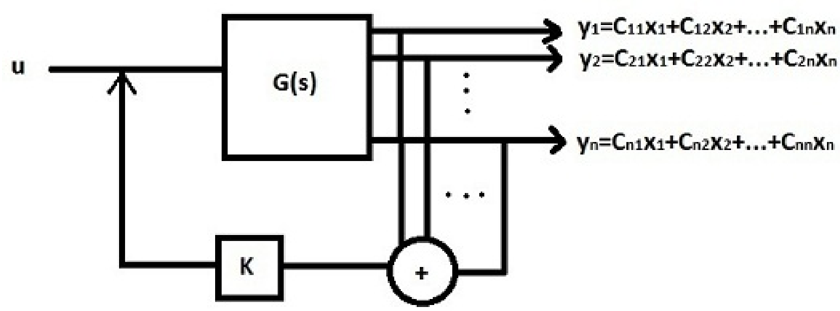
Dynamic Systems Theory - State-space Linear Systems

October 18, 2012



Output Feedback



(If C=(c_{ij}) is invertible, you can do more)

Case 1:

$$g(s)(u - k\,y) = y \Longleftrightarrow y = \frac{g(s)}{1 + k\,g(s)}\,u$$

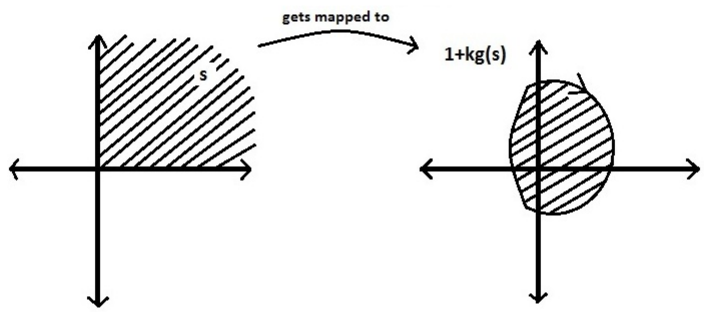
If

$$g(s) = \frac{C_n s^{n-1} + \ldots + C_2 s + C_1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

then,

$$\frac{g(s)}{1+kg(s)} = \frac{C_n s^{n-1} + \ldots + C_1}{s^n + (a_{n-1} + k\,C_n)s^{n-1} + \ldots + (a_0 + k\,C_1)}$$

We're interested in the zeros of $1 + k\,g(s)$ being in the right hand plane. We can check the image of $1 + k\,g(s)$ for $\Re(s) > 0$



It's easier to check $\{1 + k\,g(s) : s = \iota\,\omega; -\infty \leq \omega \leq \infty\}$

Nyquist Criterion Let g(s) be a scalar rational function of a complex variable $s : \sigma + \iota\,\omega$.

$$\Gamma(g) = \{u + \iota\,v : u = \Re(g(\iota\,\omega)(, v = \Im(g(\iota\,\omega)); -\infty \leq \omega \leq \infty\}$$

is called the Nyquist locus of g.

If $\Gamma(g)$ is bounded, we say the Nyquist locus encircles $(u_0 + \iota\,v_0)$, ρ times if
(a) $u_0 + \iota\,v \notin \Gamma(g)$, and
(b) $2\pi\rho$ is the net increase in the argument of $g(\iota\,\omega) - u_0 - \iota\,v_0$

Theorem: Suppose $g(s)$ has a bounded Nyquist locus. If $g(s)$ has γ poles in the r.h.p.($\Re(s) > 0$), then $\dfrac{g(s)}{1+k\,g(s)}$ has $\rho + \gamma$ poles in the r.h.p. if the point $-\dfrac{1}{k} + \iota\,0$ is not on the Nyquist locus, and $\Gamma(g)$ encircles $-\dfrac{1}{k} + \iota\,0$ times in the clockwise sense.

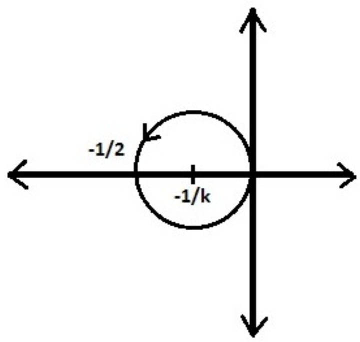
Example:

$$g(s) = \frac{1}{s-2}$$
$$\Gamma(g) = \{\frac{1}{\iota\,\omega - 2} : -\infty \leq \omega \leq \infty\}$$

Multiplying & dividing by $(-\iota\,\omega - 2)$,

$$\Gamma(g) = \{\frac{-2}{\omega^2+4} - \iota\frac{\omega}{\omega^2+4} : -\infty \leq \omega \leq \infty\}$$

The Nyquist locus looks like:



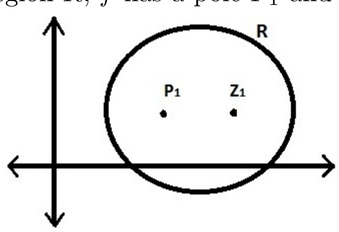
The Nyquist locus encircles $-1/k$ either

0 times if $-1/k < -1/2 \Longleftrightarrow 1/k > 1/2 \Longleftrightarrow k < 2$

or

-1 times if $-1/k > -1/2 \Longleftrightarrow 1/k < 1/2 \Longleftrightarrow k > 2$

Proof: Let f be a rational function of a complex variable s. Suppose that in a region R, f has a pole P_1 and a zero Z_1.



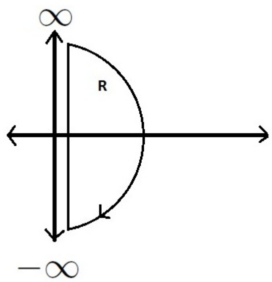
Write $f(s) = K\dfrac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_n)}$
Let s trace a tiny circle clockwise about z_1

$$s = s(\theta) = (z_1 + \varepsilon e^{\iota\omega}) \qquad 0 < \omega < \infty$$

Then,

$$\hat{f}(\theta) = \frac{K\,\varepsilon\,e^{\iota\theta}(s(\theta)-z_2)\ldots(s(\theta)-z_n)}{\varepsilon\,e^{\iota\,\theta}\ldots(s(\theta)-p)} = \frac{K}{\varepsilon}\,e^{-\iota\theta}g(\theta) \text{ and}$$

the argument decreases by 2π. More generally, the argument of f changes by $(z-p)2\pi$ as a curve is traversed clockwise around a region containing z-zeros and p-poles(counting multiplicities). Hence, as ω runs from $-\infty$ to ∞, we can think of tracing a very large "D" shaped region in the r.h.p., and the argument of $1+k\,g(s)$ changes by $2\pi\times$(no. of times origin is encircled) $= 2\pi\times\rho = 2\pi\times$(no. of times $g(s)$ encircles).



Further remarks on frequency domain stability analysis Type-m systems and the "Final Value Theorem":
Consider a signal e^t; $0 \le t \le \infty$,
$\lim_{s\to 0} s\hat{e}(s) = \int_0^\infty se(t)e^{-st}\,du$

$$= \lim_{s\to 0}\left[-e(t)\,e^{-st}|_0^\infty + \int_0^\infty e'(t)\,e^{-st}\right]$$

$$= e(0) + \int_0^\infty e'(t)\,dt$$

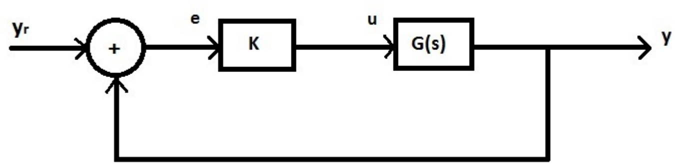
$$= e(0) + \lim_{t\to\infty} e(t) - e(\theta)$$

$$= \lim_{t\to\infty} e(t)$$

Laplace Transform Final Value Theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to\infty} s\,\hat{e}(s)$$

An Application: Tracking



$e = y_\nu - y$
$\hat{e} = \hat{y}_\nu\ = (I + G(s)\,K)\,\hat{y}_\nu$
The transfer function is $(I + G(s)\,K)^{-1}$ from y_ν to e.
Definition: A system is said to be of Type-m if it can track a polynomial input of degree m with finite, but non-zero steady state error.
Suppose,
$y_\nu(t) = C_0 + C_1\,t + \ldots + C_m\,t^m$

$\hat{y}_\nu(s) = \frac{C_0}{s} + \frac{C_1}{s^2} + \ldots + + \frac{C_m}{s^{m+1}} = \frac{1}{s^{m+1}} (C_0\, s^m + \ldots + c_m)$

Then,

$s\,\hat{e}(s) = \frac{1}{1 + k\,g(s)}$

If $g(0)$ is finite, $\lim_{s\to 0} s\,\hat{e}(s) = \infty$. The only way $s\,\hat{e}(s) \to 0$ is if $g(s)$ has $s \to 0$ pole of order$>m$ at $s = 0$.

Given,

$\dot{x} = A\,x + b$, x_0 is an equilibrium.

Solution

$\iff A\,x_0 + b = 0$

$\iff x_0 + A^{-1} b$ in the case that A is not invertible.

The equilibrium X_0 is <u>asymptotically stable</u> if the state converges X_0 for all initial conditions.

The solution to this differential equation is,

$e^{A\,t}\left(X(0) + A^{-1}\,b\right) - A^(-1)\,b$,

and the equilibrium will be asymptotically stable if $e^{A\,t} \to 0$ as $f \to \infty$.

Let $\lambda = a + ib$, and write $e^{\lambda\,t} = e^{(a+ib)\,t}$

$e^{\lambda\,t} = e^{(a+\iota\,b)\,t}$

$= e^{a\,t}\,e^{\iota\,b\,t}$

$= e^{a\,t}\,(\cos b\,t + i \sin b\,t)$

$$e^{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} t} = \begin{pmatrix} e^{\lambda\,t} & t\,e^{\lambda\,t} & \frac{t^2\,e^(3\,t)}{2} \\ 0 & e^{\lambda\,t} & t\,e^{\lambda\,t} \\ 0 & 0 & e^{\lambda\,t} \end{pmatrix}$$

In general, when there is a non-trivial Jordan block, there will be matrix entries involving terms $t^k\,e^{\lambda\,t}$ for positive integers k. Then,

$\lim_{t\to\infty} t^k\,e^{\lambda\,t} = \lim_{t\to\infty} \frac{t^k}{e^{-\lambda\,t}} = \lim_{t\to 0} \frac{k\,t^{k-1}}{-\lambda\,e^{-\lambda\,t}}$ (L'Hospital)

$= \lim_{t\to\infty} \frac{k!}{(-\lambda)^k\,e^{-\lambda\,t}} = 0$

In general, the dynamic characteristics associated with eigenvalue $\lambda = a + \iota\,b$ are,

$e^{a\,t}\,(\cos b\,t + i \sin b\,t))\;p(t)$