Dynamic Systems - State Space Control - Lecture 12 *

October 16, 2012

THEOREM: Suppose G(s) is a qxm matrix of rational functions such that the degree of denominator of each element exceeds the degree of the numerator. Then there exist constant matrices A,B and C such that,

$$G(s) = C(sI - A)^{-1}B$$

PROOF: Let $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ be the monic LCM of the denominator. Write,

$$G(s) = \frac{E_{n-1}s^{n-1} + \cdots + E_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Then,

$$A = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 I_m & -a_1 I_m & -a_2 I_m & \cdots & -a_{n-1} I_m \end{pmatrix}$$

$$B = \begin{pmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{pmatrix}$$

$$C = \begin{pmatrix} E_0 & \vdots & E_1 & \vdots & \cdots & \vdots & E_{n-1} \end{pmatrix}$$

*This work is being done by various members of the class of 2012

Standard Controllable Realization

Let p(s) be the same LCM. This time expand G(s) about $s = \infty$ to get,

$$G(s) = L_0 \frac{1}{s} + \frac{1}{s^2} + \cdots$$

Let,

$$A = \begin{pmatrix} 0_q & I_q & 0_q & \cdots & 0_q \\ 0_q & 0_q & I_q & \cdots & 0_q \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 I_q & -a_1 I_q & -a_2 I_q & \cdots & -a_{n-1} I_q \end{pmatrix}$$

$$B = \begin{pmatrix} L_0 \\ L_1 \\ \vdots \\ L_{n-1} \end{pmatrix}$$

$$C = \begin{pmatrix} I_q & \vdots & 0_q & \vdots & \cdots & \vdots & 0_q \end{pmatrix}$$

We have already seen that $C(sI - A)^{-1}B = CB\frac{1}{s} + CAB\frac{1}{s^2} + \cdots$
Since,

$$CB = L_0$$

$$CAB = \begin{pmatrix} 0_q & \vdots & I_q & \vdots & 0_q & \cdots & 0_q \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \\ \vdots \\ L_n \end{pmatrix} = L_1 \qquad CA^{n-1}B = L_{n-1}$$

$C(sI - A)^{-1}B$ and G(s) agree upto the term $L_{n-1}\frac{1}{s^n}$.
Now,

$$\det(sI - A) = \begin{vmatrix} I_n s - A_0 & 0_n & \cdots & 0_n \\ 0_n & I_n s - A_0 & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & I_n s - A_0 \end{vmatrix}$$
$$= \det(I_n s - A_0)$$

Where,

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

(Verify this!)

Since $p(A_0) = 0$ we also have $p(A) = 0$

$$p(s)G(s) = (s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)(L_0\frac{1}{s} + L_1\frac{1}{s^2} + \cdots) \quad (*)$$

The coefficient of $\frac{1}{s}$ in $(*)$ is,

$$a_0L_0 + a_1L_1 + \cdots + a_{n-1}L_{n-1} + L_n = 0$$

At the same time, since $p(A) = 0$,

$$\begin{aligned} & a_0CB + a_1CAB + \cdots + a_{n-1}CA^{n-1}B + CA^nB = 0 \\ \Rightarrow \quad & L_n = CA^{n-1}B \end{aligned}$$

This argument can be repeated due to the fact that the coefficient of $\frac{1}{s^2}$ in p(s)G(s) must also vanish and inductively it follows that $CA^kB = L_k$ for all values of k.

This is standard observable realization.

EXAMPLE: $g(s) = \frac{s+1}{s^2+1}$

The standard controllable realization is,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

<u>Check:</u>

$$(sI - A) = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2+1}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}$$

$$C(sI - A)^{-1}B = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{s^2+1} \quad = \quad \frac{s+1}{s^2+1}$$

The standard observable realization is,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} L_0 \\ L_1 \end{pmatrix}$$

Find L_k's by long division.
On dividing we obtain the quotient as $\frac{1}{s} + \frac{1}{s^2} + \cdots$

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>Check:</u>

$$C(sI - A)^{-1}b = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{s^2+1} \quad = \quad \frac{s+1}{s^2+1}$$

General Remarks on Stability:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (*)$$

$$e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

When <u>λ is real</u>, there are three cases of interest

<u>**Case 1:**</u> $\lambda < 0$

$$\begin{aligned} \lim_{t\to\infty} e^{\lambda t} &= 0 \\ \lim_{t\to\infty} te^{\lambda t} &= \lim_{t\to\infty} \frac{t}{e^{-\lambda t}} \\ &= \lim_{t\to\infty} \frac{t}{-\lambda e^{-\lambda t}} \qquad \text{(l'Hôpital)} \\ &= \lim_{t\to\infty} -\lambda e^{-\lambda t} \\ &= 0 \end{aligned}$$

$$\lambda < 0 \implies e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } t \to \infty$$

The origin $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium for the equation (*).
It is asymptotically stable in the sense that for any initial condition
$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
In general, if eigenvalues are real and < 0 then $e^{At} \to 0$ as $t \to \infty$

More generally, if the real parts of the eigenvalues of A are < 0, $e^{At} \to 0$ as $t \to \infty$.

We call a linear system $\dot{x} = Ax$ which the eigenvalues of $A < 0$ an asymptotically stable system.

Left hand plane eigenvalues of the matrix A correspond to the left half plane poles of $(sI - A)^{-1}$

Case 2: $\lambda > 0$

$$e^{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}t} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix} \rightarrow \begin{pmatrix}\infty & \infty\\ 0 & \infty\end{pmatrix} \text{ as } t \rightarrow \infty$$

In general, if A has any eigenvalue with positve real part, there are trajectories $x(t)$ such that $||x(t)|| \rightarrow \infty$ as $t \rightarrow \infty$.

Case 3: $\lambda = 0$

$$e^{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}t} = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}1 & \infty\\ 0 & 1\end{pmatrix} \text{ as } t \rightarrow \infty$$

In general, if A has any eigenvalue with positve real part, there are trajectories $x(t)$ such that $||x(t)|| \rightarrow \infty$ as $t \rightarrow \infty$.

Stability depends on the poles of the transfer function $(C(sI - A)^{-1})$

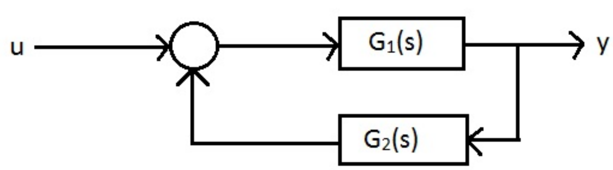
The simplest feedback control design is to use output feedback to modify system dynamics:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Let $u = Ky = KCx$ for an appropriate gain k

$$\dot{x} = (A + BkC)x \qquad \text{(Closed loop system)}$$

Is it stable?



The control input is $u + G_2 y$

$$y = G_1(u + G_2 y)$$

Solving for y,

$$y = \frac{G_1}{I - G_1 G_2} u \quad \text{or} \quad (I - G_1 G_2)^{-1} G_1 u$$

Supposing $G_2(s) \equiv K$,

$$y = \frac{G_1}{I - G_1 K} u \quad \text{or} \quad (I - G_1 K)^{-1} G_1 u$$

Suppose, $G_1(s)$ comes from a linear, time-invarient system,

$$G_1 = C(sI - A)^{-1} B$$

Then consider replacing the control with a feedback law,

$$U \to U + KCx$$

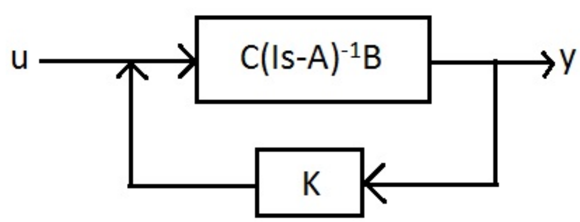
The modified dynamics are,

$$\begin{aligned} \dot{x} &= (A + BKC)x + Bu \\ y &= Cx \end{aligned}$$

The transfer function relationship is,

$$Y(s) = C(sI - A - BKC)^{-1} Bu(s)$$

How does this relate to



?

From what we saw above, the closed loop transfer function is,

$$(I - C(sI - A)^{-1} BK)^{-1} C(sI - A)^{-1} B$$

When C = I (i.e when all states are observed)

$$\begin{aligned} (I - (sI - A)^{-1} BK)^{-1} (sI - A)^{-1} B &= [(sI - A)\{\ I - (sI - A)^{-1} BK\}]^{-1} B \\ &= [(sI - A) - BK]^{-1} B \\ &= (sI - A - BK)^{-1} B \end{aligned}$$

Which is what we obtained from the time domain representation.

In this case, the poles of the system tell the stability story, and these are accessible from the system outputs.

They are zeros, $I - G(s)K$. As K varies these constitute the root locus.