# Lecture 10 *

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$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad TimeDomain$$

$$\Updownarrow$$

$$\hat{y} = C(Is - A)^{-1}B\hat{u} \qquad FrequencyDomain$$

Time-invariant case only
Recall:

$$(sI - A)^{-1} = \frac{1}{|sI - A|} \cdot adj(sI - A) \qquad (*)$$

$$adj \doteq adjoint,\ adjunct,\ adjugate$$

Now,

$$|sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

is the characteristic polynomial.

(Convince yourself that each cofactor of $sI - A$ is a polynomial of degree $< n$. )

Hence, $adj(sI - A) = E_{n-1}s^{n-1} + E_{n-2}s^{n-2} + \cdots + E_1 s + E_0$

Thus,

$$(sI - A)^{-1} = \frac{E_{n-1}s^{n-1} + \cdots + E_1 s + E_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

From (*),

$$\begin{aligned}(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)I &= E_{n-1}s^n + E_{n-2}s^{n-1} + \cdots + E_1 s^2 + E_0 s \\ &\quad - AE_{n-1}s^{n-1} - \cdots - AE_1 s - AE_0 \\ &= E_{n-1}s^n + (E_{n-2} - AE_{n-1})s^{n-1} + \cdots + (E_0 - AE_1)s - AE_0\end{aligned}$$

*This work is being done by various members of the class of 2012

We equate coefficients:

$$
\begin{aligned}
I &= E_{n-1} \\
a_{n-1}I &= E_{n-2} - AE_{n-1} \\
a_{n-2}I &= E_{n-3} - AE_{n-2} \\
&\ \vdots \\
a_1 I &= E_0 - AE_1 \\
a_0 I &= -AE_0
\end{aligned}
$$

**Theorem.** *The coefficients*

$$a_0,\ a_1,\ \cdots,\ a_{n-1}$$

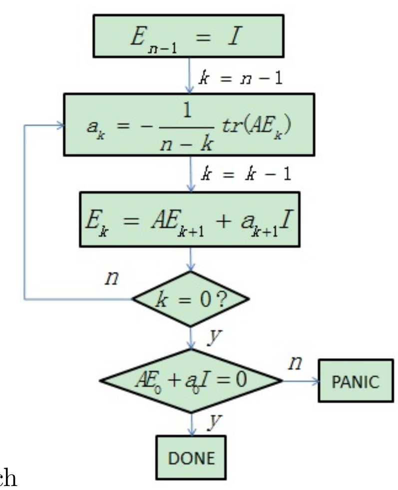
$$E_0,\ E_1,\ \cdots,\ E_{n-1}$$

*in the expression* $(sI-A)^{-1} = \frac{E_{n-1}s^{n-1}+\cdots+E_1 s+E_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$*, may be determined successfully by means of the recursive formulas:*

*(1)* $\begin{cases} E_{n-1} = I \\ E_k = AE_{k+1} + a_{k+1}I \end{cases}$

*(2)* $a_k = -\frac{1}{n-k}tr(AE_k) \quad k = n-1, n-2, \cdots, 0$

Writing the algorithm:

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*Proof.* It is really required only to show (2).

First, note:

$$
\begin{aligned}
AE_{n-1} &= AI = A \\
AE_{n-2} &= A^2E_{n-1} + a_{n-1}I = A^2 + a_{n-1}A \\
AE_{n-3} &= A^3 + a_{n-1}A^2 + a_{n-2}A \\
&\vdots \\
AE_k &= A^{n-k} + a_{n-1}A^{n-k-1} + \cdots + a_{k+1}A \\
&\vdots \\
tr(AE_k) &= tr(A^{n-k}) + a_{n-1}tr(A^{(n-k-1)}) + \cdots + a_{k+1}tr(A)
\end{aligned}
$$

Claim 1:

$$tr(A^j) = \sum_{i=1}^{n} \lambda_i^j,$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues (not necessarily distinct) of $A$.

Proof of the claim:

For any matrix $B$, $tr(B) = \sum_{i=1}^{n} \lambda_i(B)$. (If $B$ is any matrix, and $P$ is a change of basis that yields the Jordan Normal Form of $B$, call it $J_B$, then:

$$P^{-1}BP = J_B$$

$$\sum_{i=1}^{n} \lambda_i = tr(J_B) = tr(P^{-1}BP) = tr(BPP^{-1}) = tr(B))$$

If $\lambda$ is an eigenvalue of A, then $\lambda^j$ is an eigenvalue of $A^j$ (because $Ax = \lambda x \Rightarrow A^2x = \lambda Ax = \lambda^2 x$). The Claim 1 follows.

Hence,

$$tr(AE_k) = s_{n-k} + a_{n-1}s_{n-k-1} + \cdots + a_{k+1}s_1,$$

where $s_j = \sum_{i=1}^{n} \lambda_i^j$.

The theorem now follows from Newton's formula

$$s_{n-k} + a_{n-1}s_{n-k-1} + \cdots + a_{k+1}s_1 = -(n-k)a_k$$

$\square$

Newton's formula may be found in Fadeev, V.N. Computational Aspects of Linear Algebra, Dover 1959.

**Theorem.** *Consider the polynomial:*

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

*Let $\lambda_1, \cdots, \lambda_n$ be the (not necessarily distinct) roots of this polynomial, and let*

$$s_k = \sum_{i=1}^{n} \lambda_i^k$$

*Then for $k = 1, \cdots, n$,*

$$s_k + a_{n-1}s_{k-1} + \cdots + a_{n-k-1}s_1 + ka_{n-k} = 0$$

*Proof.* Associate to each root $\lambda_i$, a monic polynomial of degree n-1 which has root $\lambda_1, \cdots, \hat{\lambda}_i, \cdots, \lambda_n$, where "$\hat{\lambda}_i$" means that the i-th root is omitted from the list. Denote this polynomial by:

$$s^{n-1} + a_{n-2}^i s^{n-2} + \cdots + a_0^i = p_i(s)$$

$$\begin{aligned}
s^{n-1} + a_{n-2}^i s^{n-2} + \cdots + a_0^i &= \frac{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{s - \lambda_i} \\
&= s^{n-1} + (a_{n-1} + \lambda_i)s^{n-2} + (a_{n-2} + a_{n-1}\lambda_i + \lambda_i^2)s^{n-3} + \cdots + \\
&+ (a_2 + a_3\lambda_i + a_4\lambda_i^2 + \cdots + a_{n-1}\lambda_i^{n-3} + \lambda_i^{n-2})s \\
&+ (a_1 + a_2\lambda_i + a_3\lambda_i^2 + \cdots + a_{n-1}\lambda_i^{n-2} + \lambda_i^{n-1}) \\
&+ (a_0 + a_1\lambda_i + a_2\lambda_i^2 + \cdots + a_{n-1}\lambda_i^{n-1} + \lambda_i^{n})s^{-1} \\
&+ \cdots
\end{aligned}$$

But, of course, there are no terms with negative exponents.

Equating coefficients,

$$\begin{aligned}
a_{n-2}^i &= a_{n-1} + \lambda_i \\
a_{n-3}^i &= a_{n-2} + a_{n-1}\lambda_i + \lambda_i^2 \\
&\vdots \\
a_1^i &= a_2 + a_3\lambda_i + \cdots + a_{n-1}\lambda_i^{n-3} + \lambda_i^{n-2} \\
a_0^i &= a_1 + a_2\lambda_i + \cdots + a_{n-1}\lambda_i^{n-2} + \lambda_i^{n-1}
\end{aligned}$$

Then,

$$\begin{aligned}
\lambda_1 a_k^1 + \lambda_2 a_k^2 + \cdots + \lambda_n a_k^n &= \lambda_1(a_{k+1} + a_{k+2}\lambda_1 + \cdots + a_{n-1}\lambda_1^{n-k-2} + \lambda_1^{n-k-1}) \\
&+ \lambda_2(a_{k+1} + a_{k+2}\lambda_2 + \cdots + a_{n-1}\lambda_2^{n-k-2} + \lambda_2^{n-k-1}) \\
&\vdots \\
&+ \lambda_n(a_{k+1} + a_{k+2}\lambda_n + \cdots + a_{n-1}\lambda_n^{n-k-2} + \lambda_n^{n-k-1})
\end{aligned}$$

This last expression

$$= a_{k+1}s_1 + a_{k+2}s_2 + \cdots + a_{n-1}s_{n-k-1} + s_{n-k}$$

We know that each $a_k^i$ is the sum of products of roots in the list $\lambda_1, \cdots, \hat{\lambda}_i, \cdots, \lambda_n$. This may be written explicitly

$$\begin{aligned}
a_0^i &= (-\lambda_1)(-\lambda_2)\cdots(-\hat{\lambda}_i)\cdots(-\lambda_n) \\
&\cdots \\
a_{n-1}^i &= -\lambda_1 - \lambda_2 - \cdots - \hat{\lambda}_i - \cdots - \lambda_n \\
a_{n-3}^i &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \\
&\qquad\qquad\qquad \hookrightarrow no\ product\ with\ \lambda_i\ as\ a\ factor
\end{aligned}$$

More generally, the coefficient $a_k^i$ involves sum of products of roots taken n-k-1 at a time. From this we may explicitly write down an expression for $\lambda_1 a_k^1 + \cdots + \lambda_n a_k^n$

E.g.

$$\begin{aligned}
&\lambda_1 a_{n-2}^1 + \lambda_2 a_{n-2}^2 + \cdots + \lambda_n a_{n-2}^n \\
&= \lambda_1(-\lambda_2 - \lambda_3 - \cdots - \lambda_n) + \lambda_2(-\lambda_1 - \lambda_3 - \lambda_4 - \cdots - \lambda_n) \\
&+ \cdots + \lambda_n(-\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1}) \\
&= -2a_{n-2}
\end{aligned}$$

Next,

$$\begin{aligned}
&\lambda_1 a_{n-3}^1 + \lambda_2 a_{n-3}^2 + \cdots + \lambda_n a_{n-3}^n \\
&= \lambda_1(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \cdots - \lambda_{n-1}\lambda_n) + \lambda_2(\lambda_1\lambda_3 + \lambda_1\lambda_4 + \cdots - \lambda_{n-1}\lambda_n) + \\
&+ \cdots + \lambda_n(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots - \lambda_{n-2}\lambda_{n-1}) \\
&= -3a_{n-3}
\end{aligned}$$

This pattern persists, and in general,

$$\lambda_1 a_{n-k}^1 + \cdots + \lambda_n a_{n-k}^n = -k a_{n-k}$$

and this proves the theorem

$\square$

The results presented in today's lecture show how to get

$$C(sI - A)^{-1}B$$

given A, B, C. i.e.

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right. \Leftrightarrow \hat{y} = C(sI - A)^{-1}B\hat{u}$$

Problem to think about: Given $\hat{y} = G(s)\hat{u}$, where $G(s)$ is a matrix of proper rational functions, find $A,\ B,\ C$.