Intuitive Beliefs*

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Abstract

An agent’s intuitive beliefs over a state space are modelled as the output of a network of associations between events, where the network is understood to be shaped by past experience. The model is shown to accommodate a wide range of findings in psychology. The intersection with the Bayesian model is characterized.

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1 Introduction

Investors often describe gut-feelings and intuition as important factors in decision-making after having collected and analyzed their data. The importance of such factors for real world decision-making was underscored by Keynes’ notion of animal spirits (Keynes 1936) and has been discussed in the management literature (Huang and Pearce 2015, Huang 2018). We maintain that incomplete deliberation must be the norm rather than the exception for agents with limited cognitive resources living in a very complex world. We find it compelling that the gaps in judgement left by incomplete deliberation are generally filled by one’s gut feelings, and therefore that intuitive judgements have a possibly significant role to play in economic behavior.

This paper seeks a formal theory of intuition. We present, first, our articulation of what intuition is, and second, the mathematical structure we use to capture it:

We take inspiration from the classic notion of association in psychology and philosophy to provide a conceptual basis for the notion of intuition.\footnote{The philosophical school of Associationism, which dates back to the writings of Locke and Hume and served as the foundation of behavioral psychology in the early 20th century, recognized associations as the most basic function of the mind. The Associationists sought to reduce all mental life to associations, an endeavor that survived up until the cognitive revolution in the mid 20th century.} If one hears the word “red”, one might involuntarily think of a traffic signal. If one learns of someone’s vocation, ethnicity, religion, etc., one may involuntarily find the image of a stereotype appear in their mind. Associations are such in-built connections between mental images of events, whereby the activation
of one image activates (or inhibits) the other. Philosophers and psychologists have extensively elaborated on how associative connections can be formed through frequent or salient pairing of events and through similarities, and can be strengthened (reinforcement) or weakened (counter-conditioning or decay) over time. A central observation is that the triggering of associations is an involuntary process that entails no cognitive effort. In this regard, associative thinking is fundamentally different from deliberative reasoning. We conceptualize intuition in terms of the triggering of a chain of associations.

To formulate a mathematical model of intuitive beliefs, we loosely take inspiration from neural networks in cognitive science. We model the agent as possessing an underlying network of associations between events, where the network is understood to be formed by past experience. Triggering a node in the network through some observation leads it to send signals to other nodes, in turn triggering them, and so forth. The agent’s intuitive judgement is determined by the signal output at relevant nodes in the network. In our model, the agent’s belief about an event after having observed some event is a function of the signal output of an underlying network of associations between events.

We analyze whether intuitive beliefs can be additive, and whether they can satisfy Bayes’ rule. We find that, if there is sufficient richness in the beliefs, then additivity and Bayesian updating each require beliefs satisfy statistical independence for events that are deemed possible, that is, if the belief about the joint realization of variables is not zero then it must equal the product of the marginals. The results suggest that standard properties of probability measures are more likely to hold in intuitive judgements when
the uncertainty faced by the agent is perceived to have a relatively simple structure.

As an application of our model, we consider the classic findings in psychology that show that people’s judgements under uncertainty do not conform to the standard Bayesian model of beliefs: people’s beliefs neither respect the axioms of probability theory nor are they updated in accordance with Bayes’ Rule. Indeed, the prevalent paradigm in the psychology literature on beliefs is based on the broad idea that people’s beliefs reflect intuitive judgements, rather than judgements arrived at through deliberation. We confirm that much of the evidence is consistent with our model.

We present our model in Sections 2 and 3, describe some of its properties in Section 4, and show in Sections 5 and 6 that it can accommodate well-known findings in psychology, while also highlighting the conceptual differences from the Heuristic and Biases program of Kahneman and Tversky in the literature review in Section 7. Section 8 concludes. Proofs are contained in the appendix.

2 Primitives and Illustration

To illustrate our model we consider an individual (“Steve”) randomly selected from North America and about whom there are two sources of uncertainty: whether he is a librarian or salesperson, and his degree of introversion.\footnote{This evidence is reviewed in Tversky and Kahneman (1974), and supporting field evidence in economics can be found in DeBondt and Thaler (1985), Suetens et al (2016), and Ahmed and Safdar (2016), among others.}

\footnote{This is a variation on Kahneman and Tversky’s “Steve the librarian” experiment.}


2.1 States

The set of sources or elements of uncertainty is an abstract set $\Gamma$ with cardinality $0 < N < \infty$ and generic elements $i, j, k, \ldots$. The abstract set $\Omega_i$, with generic elements $x_i, y_i, z_i, \ldots$, consists of all possible realizations of source $i$, and is referred to as the elementary state space for source $i$. To avoid technicalities, we assume that each $\Omega_i$ is finite. The (full) state space is the product space given by

$$\Omega := \prod_{i \in \Gamma} \Omega_i,$$

with generic element $x = (x_1, \ldots, x_N)$.

In the case of Steve, the sources of uncertainty are:

$$\Gamma = \{vocation, personality\},$$

and the elementary state spaces could, for instance, be:

$$\Omega_v = \{\text{librarian, salesperson}\}$$

$$\Omega_p = \{\text{shy, normal, gregarious}\},$$

and the full state space is given by:

$$\Omega = \Omega_v \times \Omega_p.$$  

For instance, “Steve is an extremely introverted librarian” is a state of the world.

2.2 Events

An elementary event in source $i$ is a subset of $\Omega_i$, generically denoted by $x_i, y_i, z_i, \ldots$. Fix some algebra of elementary events $\Sigma_i \subset 2^{\Omega_i}$ – for simplicity let
us take this to be the power set, \( \Sigma_i = 2^{\Omega_i} \). An event in the full state space is a vector of elementary events:

\[
\Sigma = \prod_{i \in \Gamma} \Sigma_i.
\]

A generic event is given by \( x = (x_i, x_j, \ldots, x_k) \).\(^4\)

To illustrate, consider the information that “Steve is not gregarious”. This can be viewed as the following elementary event in the “personality” source:

\[ x_p = \{ \text{shy, normal} \} \in \Sigma_p. \]

The corresponding event (where there is no information about vocation) is

\[ (\Omega_v, x_p) \in \Sigma_v \times \Sigma_p. \]

### 2.3 Beliefs

For any event \( z \in \Sigma \), consider the agent’s conditional beliefs \( p(\cdot|z) \) over \((\Omega, \Sigma)\). Formally, a belief conditional on \( x \in \Sigma \) is a function that assigns \( p(x|z) \in [0, 1] \) for each event \( x \in \Sigma \) and satisfies:

1. \( p(\Omega|z) = 1 \),
2. \( p(x|z) = 0 \) if \( x_i = \phi \) for some \( i \).
3. \( p(x \cap z|z) = p(x|z) \).

\(^4\)Note that an event would normally be defined as a subset of \( \Omega \) whereas we define an event as a vector of elementary events, which is a special kind of subset of \( \Omega \). This will be convenient for our purposes as our network will have a node for every elementary event. The extension to more general events and corresponding networks is left for future research.
The first condition states that regardless of the information, the agent is certain that some state is true. The second condition states that any event which specifies some empty event is deemed impossible. The third condition captures the idea that the agent understands that her information \( z \) rules out the possibility of states outside \( z \), and so when evaluating the likelihood of event \( x \) she effectively evaluates the likelihood of \( x \cap z \). We will consider both additive and nonadditive beliefs in the sequel.

The set of all conditional beliefs is given by:

\[
p = \{ p(\cdot|z) : z \in \Sigma \text{ s.t. } p(z|\Omega) > 0 \}.
\]

We refer to this set simply as the agent’s beliefs. The prior is given by \( p(\cdot|\Omega) \).

One can imagine natural restrictions across conditional beliefs, but we postpone imposing restrictions until needed in the sequel.

**Notation:** For any elementary event \( x_i \in \Sigma_i \), we abuse notation and denote the full event \( x_i \Omega_{\neg i} \in \Sigma \) by

\[
x_i := x_i \Omega_{\neg i}.
\]

Moreover, for any pair of events \( x, z \in \Sigma \) we define set inclusion by pair-wise set inclusion:

\[
x \subset z \iff x_i \subset z_i \text{ for all } i.
\]

### 2.4 Illustration of Model

To preview our model, suppose an agent (“she”) is given the information that Steve was selected randomly from North America, and is asked for her believed likelihood that Steve is a shy librarian. Dropping the braces for
singleton events throughout this subsection to ease exposition, and using \( l \) to denote “librarian” and \( s \) to denote “shy”, we write her belief as:

\[
p(l, s|\Omega_v, \Omega_p).
\]

We model this agent as possessing a network of associations. Each elementary event defines a node in the network. Between any two nodes \( x_i, x_j \) there are two independent directed links with weights,

\[
a(x_i|x_j) \text{ and } a(x_j|x_i),
\]

that run from \( x_j \) to \( x_i \), and from \( x_i \) to \( x_j \), respectively.

The agent is evaluating the likelihood of the event

\[
x = (l, s).
\]

When she is presented with her prior information \( \Omega = (\Omega_v, \Omega_p) \), the nodes \( \Omega_v \) and \( \Omega_p \) are triggered, and these then directly trigger the librarian and introvert nodes, generating signals with respective strengths

\[
a(l|\Omega_v), \ a(l|\Omega_p),
\]
\[
a(s|\Omega_v), \ a(s|\Omega_p).
\]

Imagine here and below that each signal is sent sequentially.

However, the librarian node is also connected to the introvert node, and sequentially sends it signals of strength:

\[
\pi(s|l)a(l|\Omega_v),
\]
\[
\pi(s|l)a(l|\Omega_p).
\]
Therefore, the introvert node receives signals not only directly from $\Omega_v$ and $\Omega_p$ but also indirectly through the librarian node. Similarly, the librarian node receives indirect signals through the introvert node:

$$
\pi(l | s) a(s | \Omega_v),
$$

$$
\pi(l | s) a(s | \Omega_p).
$$

We permit secondary signals $\pi$ to be of different strength than the primary signals $a$. It would be natural to think that any node is that is triggered initiates a signal of some constant strength, that is the subsequent part that reaches the next node. However our applications do not require us to make any assumptions on $a$, $\pi$ except that they take values in $\mathbb{R}_+ \cup \{\infty\}$.

The theoretical framework for our theory is given by beliefs that are an increasing function of all direct and indirect signals:

$$
p(l, s | \Omega_v, \Omega_p) = f \left( a(l | \Omega_v), \pi(l | s) a(s | \Omega_v), a(l | \Omega_p), \pi(l | s) a(s | \Omega_p), a(s | \Omega_v), \pi(s | l) a(l | \Omega_v), a(s | \Omega_p), \pi(s | l) a(l | \Omega_p) \right).
$$

A natural model would be where beliefs are some increasing function $f$ of the sum of all these signals, and it would be natural to consider the familiar logistic function $f(x) = \frac{1}{1 + \exp(-x)}$ to ensure that $p$ takes values between 0 and 1. However, logistic functions and other alternatives do not prove to be tractable for subsequent analysis. We provide a general tractable formulation, an example of which would be:

$$
p(l, s | \Omega_v, \Omega_p)
= \exp \left[ - \left\{ a(l | \Omega_v)^{-1} + \pi(l | s)^{-1} a(s | \Omega_v)^{-1} + a(l | \Omega_p)^{-1} + \pi(l | s)^{-1} a(s | \Omega_p)^{-1} + a(s | \Omega_v)^{-1} + \pi(s | l)^{-1} a(l | \Omega_v)^{-1} + a(s | \Omega_p)^{-1} + \pi(s | l)^{-1} a(l | \Omega_p)^{-1} \right\} \right].
$$
That is, the reciprocal of direct and indirect signals are summed, and then passed through the reciprocal of an exponential function to determine the belief. The use of the reciprocal of the exponential function is to ensure that \( p \) lies in the unit interval. The use of the reciprocal of the signals is to ensure that \( p \) is increasing in the signals. We show in the sequel how the expression can be written more succinctly.

To illustrate the model further, suppose the agent receives the information that “Steve is shy”, corresponding to the event

\[(\Omega_v, s) \in \Sigma_v \times \Sigma_p.\]

She now evaluates

\[p(l, s|\Omega_v, s).\]

Note that in evaluating the event (librarian, shy) there is now no uncertainty about the agent’s personality. We say that personality is now an irrelevant source of uncertainty. The model requires that an infinitely large signal is sent to the shy node, resulting in

\[a(s|\Omega_v) = a(s|s) = \infty.\]

Observe that in the above expression for beliefs, by taking \( \infty^{-1} = 0 \), the infinite output at the shy node causes some of the terms to drop out of the expression, leaving:

\[p(l, s|\Omega_v, s) = \exp[-a(l|\Omega_v)^{-1} + a(l|s)^{-1}].\]

The belief is effectively determined by only the direct signals sent to the librarian node from the \( \Omega_v \) node and introvert node.
3 Intuitive Beliefs

We define a network in terms of the weights between ordered pairs of nodes. The associative network we consider will have elementary events as its nodes, and the weight between each ordered pair of nodes can be different depending on whether it is triggered directly by observation or indirectly through some other node.

3.1 Associative Network

If the agent receives information $z = (z_1, \ldots, z_N)$, then each node $z_i$ is triggered and (potentially) sends a signal directly to all nodes in the network. We refer to these as primary signals. The primary signal sent to node $x_i$ is given by $a(x_i|z)$, where we have allowed for generality in how the signals from triggered nodes combine to produce output at node $x_i$.

Definition 1 (Primary Signal) A primary signal function $a$ is a function that maps each $(x, z) \in \Sigma \times \Sigma$ and $i \in \Gamma$ to some

$$a(x_i|z) \in \mathbb{R}_+ \cup \{\infty\},$$

and satisfies (i) $a(\phi_i|z) = 0$ and (ii) $a(x_i|z) = \infty$ whenever $x_i \supset z_i$.

The primary signal to any node $\phi_i$ corresponding to an empty elementary event is always 0. In contrast, if the agent is informed that an elementary event $z_i$ in source $i$ is true, then an infinitely large primary signal is sent to any larger event $x_i \supset z_i$. For instance, if the agent is informed that “there is rain”, then the strongest possible signal is sent to the larger elementary
event that “there is precipitation”. Note that \( a(\Omega_i | z) = \infty \). Moreover, it is possible that \( a(x_i | z) = \infty \) when \( x_i \subset z \). Intuitively, if an agent considers it impossible for precipitation to take any form other than rain, then being told that there is precipitation will cause an infinitely large signal to be sent to “rain only”, and presumably also to any elementary event that contains “rain” in it (such as “snow or rain”).

When a node receives a primary signal from nodes triggered by information, it can (potentially) send a secondary signal to all nodes in the network.

**Definition 2 (Secondary Signal)** A secondary signal function \( \overline{\alpha} \) is a function that maps each \( x \in \Sigma \) and \( i, j \in \Gamma \) to some

\[
\overline{\alpha}(x_i | x_j) \in \mathbb{R}_+ \cup \{\infty\},
\]

and satisfies \( \overline{\alpha}(x_i | x_i) = 1 \).

The requirement that “\( \overline{\alpha}(x_i | x_i) = 1 \)” plays an accounting role, enabling us to compactly write the representation we introduce shortly.

**Definition 3 (Network)** An associative network is a tuple \((a, \overline{\alpha})\) consisting of a primary signal function \(a\) and a secondary signal function \(\overline{\alpha}\).

Suppose the agent is considering the likelihood of event \( x \) given information \( z \) and suppose \( x \subset z \). It may well be that the information \( z \) completely resolves uncertainty about some sources of information. For instance, when we were told that Steve is moderately introverted but not given any information about his vocation, then there was no uncertainty about the personality trait but there remained uncertainty about the vocation. We say that \( i \) is an
irrelevant source of uncertainty if there is no uncertainty remaining. In our model, this happens if the information $z$ sends an infinitely large signal to the $i$-elementary event $x_i$, that is, $a(x_i|z) = \infty$. Similarly, we define the set of relevant sources by the sources for which the uncertainty is not resolved by the information:

$$\Gamma(x|z) = \{i \in \Gamma : a(x_i|z) < \infty\}.$$

### 3.2 Model

To ease notation, let the reciprocals of the signals be given by:

$$r := a^{-1} \text{ and } \bar{r} := \pi^{-1}.$$ 

Adopt the convention that $0 \times \infty = \infty$ and that the sum over an empty set is zero, $\sum_{i \in \emptyset} = 0$. Our model is given by:

**Definition 4 (Intuitive Beliefs)** An Intuitive Belief representation for $p$ is an associative network $(a, \pi)$ such that for any $x, z \in \Sigma$ s.t. $x \subset z$,

$$p(x|z) = \exp \left[ - \sum_{i \in \Gamma(x|z)} \sum_{j \in \Gamma(x|z)} \bar{r}(x_i|x_j) r(x_j|z) \right].$$

The use of the reciprocal of the exponential ensures that $p$ takes values in the unit interval. The use of the reciprocal of the signals then ensures that $p$ is increasing in each of these signals. It should be noted that we state the representation only for nested events $x \subset z$. By definition, beliefs satisfy $p(x \cap z|z) = p(x|z)$, and so the representation extended to non-nested events
simply replaces \( x \) with \( x \cap z \) throughout in the right-hand side expression.\(^5\)

To understand the model, suppose the agent is considering the likelihood \( p(x|z) \) of event \( x \) given information \( z \). The likelihood \( p(x|z) \) will be a function of the total signal output at each of the nodes \( x_1, \ldots, x_N \). Consider \( x_i \). If \( i \) is an irrelevant source of uncertainty \( i \notin \Gamma(x|z) \) then, by definition, it receives an infinitely large primary signal

\[
a(x_i|z) = \infty.
\]

On the other hand if \( i \in \Gamma(x|z) \), then node \( x_i \) receives a finite primary signal \( a(x_i|z) < \infty \). Since \( \overline{a}(x_i|x_i) = 1 \) by definition, we can write this primary signal as:

\[
\overline{a}(x_i|x_i)a(x_i|z).
\]

This primary signal received at \( x_i \) is followed by secondary signals from all other nodes in \( x \).\(^6\) The total strength of the secondary signal from any \( x_j \) into \( x_i \) is given by

\[
\overline{a}(x_i|x_j)a(x_j|z).
\]

The likelihood \( p(x|z) \) is determined as follows. For each \( i \), the reciprocal of the signals entering \( x_i \) are summed, and then the sum of these quantities

\[
p(x|z) = \exp \left[ - \sum_{i \in \Gamma(x|z)} \sum_{j \in \Gamma(x|z)} \overline{a}(x_i \cap z_i|x_j \cap z_j) \rho(x_i \cap z_i|x_j \cap z_j) \right].
\]

\(^5\)That is, for any \( x, z \),

\[
p(x|z) = \exp \left[ - \sum_{i \in \Gamma(x|z)} \sum_{j \in \Gamma(x|z)} \overline{a}(x_i \cap z_i|x_j \cap z_j) \rho(x_i \cap z_i|x_j \cap z_j) \right].
\]

\(^6\)One can imagine that secondary signals could be received from all nodes in the network, and not just from \( x_1, \ldots, x_N \). We abstract from this, imagining that signals running between the direct considered nodes \( x_1, \ldots, x_N, z_1, \ldots, z_N \) swamp signals from nodes that are not being considered directly.
across all $i$ are summed and passed through the reciprocal of an exponential function to yield a number $0 \leq p(x|z) \leq 1$. More formally:

$$p(x|z) = \exp \left[ - \sum_{l \in \Gamma(x|z)} r(x_l|z) + \sum_{i \in \Gamma(x|z)} \sum_{j \in \Gamma} \tau(x_i|x_j) r(x_j|z) \right].$$

Observe first that $r(x_i|z) = 0$ for all irrelevant $l \not\in \Gamma(x|z)$, since $r(x_i|z) = a(x_i|z)^{-1} = \infty^{-1} = 0$. So we can drop the first term in the expression. Observe furthermore that, in the second term, although we have taken a sum across all $j \in \Gamma$, it is the case that $\tau(x_i|x_j) r(x_j|z) = 0$ whenever $j \not\in \Gamma(x|z)$ (since $r(x_j|z) = 0$, as before). Consequently we can replace $\sum_{j \in \Gamma}$ with $\sum_{j \in \Gamma(x|z)}$. Then we get exactly the functional form of an Intuitive belief representation.

### 3.3 A Special Case

The model is very general and as such calls for some natural properties. For instance, it would be natural that beliefs about the likelihood of an event should not change if the event is expanded to include impossible states. In Appendix A.1 we outline such “regularity properties” and use them to prove our main results in Section 4. However, it is also instructive to consider a concrete version of the model that subsumes natural regularity properties.

Suppose that there are primary $a(x_i|y_j) \in \mathbb{R}_+ \cup \{\infty\}$ and secondary $\tau(x_i|y_j) \in \mathbb{R}_+ \cup \{\infty\}$ signal strengths between every ordered pair of elementary states $(x_i, y_j) \in \Omega_i \times \Omega_j$ for each pair of distinct sources $i, j \in \Gamma$. We define an associative network using these.

**Definition 5 (Structured Network)** An associative network $(a, \tau)$ is struc-
tured if it satisfies for each $x_i, x_j, z$ s.t. $i \neq j$ and $x_j \subset z_j$,

$$
\overline{a}(x_i|x_j) = \sum_{x_i \in x_i} \sum_{x_j \in x_j} \overline{a}(x_i|x_j) \text{ and } a(x_j|z) = \frac{1}{\sum_{x_j \in x_j \setminus x_j} \sum_{k \in \Gamma} \sum_{z_k \in z_k} a(x_j|z_k)}.
$$

The secondary signal $\overline{a}(x_i|x_j)$ between two events $x_i, x_j$ is simply the sum of signals sent from each elementary state in $x_j$ to each elementary state in $x_i$. This is a natural specification. It is less obvious, however, to construct a primary signal $a(x_j|z)$. Define

$$
A(x_j|z) : = \sum_{x_j \in x_j} \sum_{k \in \Gamma} \sum_{z_k \in z_k} a(x_j|z_k),
$$

$$
A(z_j|z) : = \sum_{z_j \in z_j} \sum_{k \in \Gamma} \sum_{z_k \in z_k} a(z_j|z_k).
$$

The quantity $A(x_j|z)$ is simply the sum of signals sent from each $z_k \in z_k$ and $k \in \Gamma$, to each $x_j \in x_j$. Note that the elementary event $z_j$ is certain under information $z$. The quantity $A(z_j|z)$ – the sum of signals sent from each $z_k \in z_k$ and $k \in \Gamma$, to each $z_j \in z_j$ – is the total signal that is sent to the certain elementary event $z_j$. But by definition of a primary signal we need that any certain elementary event should receive an infinitely large primary signal, $a(z_j|z) = \infty$. It would also be desirable if the primary signal $a(x_j|z)$ is increasing in the size of $x_j$, and decreasing in the size of $z$, as these would lead to the natural monotonicity properties that larger events are deemed more likely, and the likelihood of a given event is lower when the information allows for more possibilities.

These necessary and desirable properties hold in the functional form: \footnote{A natural alternative functional form is $a(x_j|z) = \frac{A(x_j|z)}{A(z_j|z) - A(x_j|z)}$, where $a(x_j|z)$ is}

$$
a(x_j|z) = \frac{1}{\sum_{x_j \in x_j \setminus x_j} \sum_{k \in \Gamma} \sum_{z_k \in z_k} a(x_j|z_k)} = \frac{1}{A(z_j|z) - A(x_j|z)}.
$$
Moreover, this functional form offers tractability, since in the representation we get additivity of \( r(x_j|z) \): \( r(x_j|z) := a(x_j|z)^{-1} = \sum_{k \in \Gamma} \sum_{x_j \in z_j \setminus x_j} \sum_{z_k \in z_k} a(x_j|z_k) \).

We use the term *structured Intuitive belief* for an Intuitive belief that is represented by a structured network. We conclude this section by confirming that:

**Proposition 1** If \( p \) is a structured Intuitive belief, then it satisfies:

(i) *(Event Monotonicity)* If \( x \subset x' \) then \( p(x|z) \leq p(x'|z') \).

(ii) *(Information Monotonicity)* If \( z \subset z' \) then \( p(x|z) \geq p(x|z') \).

## 4 Properties of the Model

### 4.1 Associations

We say that \( x \) is *positively associated* with information \( z \) if

\[
p(x|z) \geq p(x|\Omega),
\]

that is, if observing \( z \) increases the agent’s belief relative to the prior. *Negative associations* are similarly defined in terms of decreases in belief relative to the prior. Thus, in our model, associations are defined relative to the prior, and indeed, correspond to updating behavior.

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8This is proved in Lemma 1 (Appendix A.2).
In terms of the model, if the total signal output at relevant nodes is higher after the information than before, then the agent exhibits a positive association. An expression of this can be found in the updating rule. For simplicity, consider $x, z$ such that $\Gamma(x|z) = \Gamma(x|\Omega)$. While the Bayesian updating rule takes the form $p(x|z) = p(x|\Omega) \frac{p(x|z)}{p(z|\Omega)}$, the expression for our associative update rule is

$$p(x|z) = p(x|\Omega) \exp \left[ - \sum_{i,j \in \Gamma(x|z)} r(x_i|x_j)(r(x_j|z) - r(x_j|\Omega)) \right].$$

The term $[r(x_j|z_k) - r(x_j|\Omega_k)]$ is the change in the direct signal into $x_j$ after the receipt of information $z_k$. If this change is sufficiently positive at sufficiently many nodes, the association is positive, and the posterior is higher than the prior.

### 4.2 History-Independence

If an agent learns two events then she learns their intersection. Therefore any history of events is described as a nested sequence of events $z_1 \supset z_2 \supset z_3 \ldots \supset z_n$. If this sequence terminates at $z_n$ then according to the model the agent holds beliefs $p(x|z_n)$. These beliefs rely only on the associations triggered by $z_n$ and in particular are independent of how the agent arrived at $z_n$. Therefore the model exhibits history independence.

### 4.3 Bayesian Intuitive Beliefs

We define $\Gamma(x|z) \subset \Gamma(x|\Omega)$ since $i \in \Gamma(x|z)$ means $x_i \not\subset z_i$ which implies $x_i \not\subset \Omega_i$ and in turn $i \in \Gamma(x|\Omega)$. 

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9In general, $\Gamma(x|z) \subset \Gamma(x|\Omega)$ since $i \in \Gamma(x|z)$ means $x_i \not\subset z_i$ which implies $x_i \not\subset \Omega_i$ and in turn $i \in \Gamma(x|\Omega)$. 

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Definition 6 (Bayesian Beliefs) \( p \) is (Non-Additive) Bayesian if for any \( x, z \in \Sigma \),
\[
p(x|z) = \frac{p(x \cap z|\Omega)}{p(z|\Omega)}.
\]

The term “Bayesian” is usually used for additive probability measures, but we use the term to describe generically non-additive beliefs that satisfy the Bayesian conditioning formula.

A natural question is whether intuitive beliefs can be Bayesian, and if so, what characterizes the intersection of the two classes of models. The main result of this section characterizes the intersection of the class of Bayesian beliefs and the class of structured Intuitive beliefs (but see Appendix A.2 for a much more general result).

Say that beliefs satisfy richness if (a) \( \Gamma \) is not a singleton, and for each \( i \in \Gamma \), (b) \( p(\{x_i, x'_i\}|\Omega) < 1 \) for any \( x_i, x'_i \in \Omega_i \). That is, there is more than one source of uncertainty, and any binary elementary event is uncertain (or impossible).\(^{10}\)

We now show that given richness, the class of Bayesian structured Intuitive beliefs are a strict subset of the class of beliefs that satisfy a restricted form of Statistical Independence.

Theorem 1 Suppose \( p \) is a structured Intuitive Belief that satisfies richness. Then \( p \) is Bayesian if and only if it satisfies

(i) (Positive Statistical Independence): for all \( x, z \in \Sigma \) s.t. \( p(x|z) > 0 \),
\[
p(x|z) = \prod_{i \in \Gamma} p(x_i|z).
\]

\(^{10}\)Since \( p(\Omega_i|\Omega) = 1 \), richness implies that \( \Omega_i \) is not a singleton.
(ii) (Marginal Conditioning) for all \(x, z\) s.t. \(p(x|z) > 0\),

\[
p(x|z) = \frac{p(x \cap z|\Omega)}{p(z|\Omega)}.
\]

Positive Statistical Independence requires that beliefs about possible events are a product of their marginals. Since this is only a restriction for events that are deemed possible, Positive Statistical Independence permits the possibility that all marginals are positive but the joint belief is zero. Marginal Conditioning embodies a form of Bayesian conditioning of marginal beliefs augmented with the property that the updating of marginals on source \(i\) is independent of the information about other sources.\(^{11}\) It is noteworthy that while Bayesian updating is a relationship among all conditional beliefs, it imposes a strong restriction (namely Positive Statistical Independence) on each conditional belief. This is a reflection of the fact that in the world of Intuitive beliefs, conditional preferences are not independent of each other: conditional beliefs \(p(\cdot|z)\) for each \(z\) all share the same function \(\pi\).

One upshot of this result is that it is in fact possible for Intuitive beliefs to be Bayesian, albeit only when beliefs have a particularly simple structure. Presumably if the uncertainty perceived by the agent is in some sense complex, then she is likely to be Bayesian.

The theorem requires that there exist at least two sources of uncertainty. When there is only one source of uncertainty, the model takes the form

\[^{11}\text{It is straightforward to show how this condition along with Positive Statistical Independence yields Bayesian updating of possible events: } p(x|z) = \prod_{i \in \Gamma} p(x_i|z) = \prod_{i \in \Gamma} \frac{p(x_i \cap z_i|\Omega)}{p(z_i|\Omega)} = \prod_{i \in \Gamma} \frac{p(x_i \cap z_i|\Omega)}{p(z_i|\Omega)} = \frac{p(x \cap z|\Omega)}{p(z|\Omega)}.\]
\[ p(x_i|z_i) = \exp[-r(x_i|z_i)] \] and clearly lacks any kind of structure. Indeed, Bayesian Intuitive beliefs are simply Bayesian beliefs when there is only one source of uncertainty.

### 4.4 Additive Intuitive Beliefs

Next turn to the characterization of additivity. The usual definition of additivity does not apply in our setting because \( p \) is defined on vectors of events, that is, \( \Sigma = \prod_{i \in \Gamma} \Sigma_i \). Here \( \Sigma_i \) is an algebra and not \( \Sigma \). Consequently, the union \( x \cup x' \) of two events \( x \) and \( x' \) in \( \Sigma \) may not belong to \( \Sigma \). However, if we have events of the form \( x; x_{-i} \) and \( x'; x_{-i} \), that is, events that differ only on one source, then there exists a point-wise union of these events in \( \Sigma \), denoted \( x_i \cup x'_i; x_{-i} \). We exploit this to define:

**Definition 7 (Additivity)** \( p \) satisfies **Source Additivity** if all \( x, x', z \in \Sigma \) and \( i \in \Gamma \) s.t. \( x_i \cap x'_i = \phi \),

\[
p(x_i \cup x'_i; x_{-i}|z) = p(x_i; x_{-i}|z) + p(x'_i; x_{-i}|z).
\]

\( p \) satisfies **Marginal Additivity** if all \( x, x', z \in \Sigma \) and \( i \in \Gamma \) s.t. \( x_i \cap x'_i = \phi \),

\[
p(x_i \cup x'_i|z) = p(x_i|z) + p(x'_i|z).
\]

Marginal Additivity is the generalization of Source Additivity that applies only to marginals. Our next result requires a richness condition on beliefs. Say that \( p \) satisfies **richness*** if there are at least 3 distinct sources, and for any \( x_i \) s.t. \( 0 < p(x_i|\Omega_i) < 1 \) and any \( z, z' \) s.t. \( x_i \subset z_i = z'_i \),

\[
z \nsubseteq z' \implies p(x_i|z) \neq p(x_i|z').
\]

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That is, for any $x_i$ that is possible and uncertain and is contained by current information, decreasing the available information always changes the conditional belief about $x_i$.

Richness∗ is required only for the sufficiency part of the following result.

**Theorem 2** Consider a structured Intuitive Belief $p$ that satisfies richness∗. Then $p$ satisfies Source Additivity iff it satisfies Marginal Additivity and Positive Statistical Independence.

The result tells us that Source Additivity also imposes Positive Statistical Independence on Intuitive beliefs.

Put together we see that within the context of Intuitive beliefs, the class of Bayesian beliefs and the class of Source Additive beliefs are overlapping and distinct, which is implied by the fact that the updating and additivity restrictions implied by each are not mutually exclusive. Moreover, both are (strict) subsets of the class of beliefs satisfying Positive Statistical Independence. A (potentially testable) implication is that if beliefs do not satisfy Positive Statistical Independence, then non-Bayesian updating and non-Additivity must occur simultaneously.

### 5 Updating Biases

Much of what is known about beliefs comes from the Heuristics and Biases program of Kahneman and Tversky (henceforth KT). Their Representativeness heuristic specifically organizes the evidence on how people update their beliefs. We study the updating properties of the Intuitive Beliefs model in
light of this evidence. The evidence on non-additivity and non-monotonicity of beliefs addressed in Section 6.

5.0.1 Base Rate Neglect vs Conservatism

In some situations, subjects in experiments exhibit base-rate neglect (they respond to new information to the extent of neglecting base-rates and consequently overshoot the Bayesian update) whereas in others they exhibit conservatism (they appear not to respond enough and undershoot the Bayesian update). Each of these is illustrated in the following examples, based on KT and Edwards (1968), respectively:

- Base-rate neglect: You are told Joe is shy and likes to read. Is it more likely Joe is a farmer or librarian? People guess librarian even though there are far more farmers than librarians, thus seeming to neglect the base rate. KT suggest that this is because the description makes Joe more similar to the prototypical librarian than the prototypical farmer.

- Conservatism: Urn $A$ contains 700 red and 300 blue chips, and urn $B$ contains 300 red and 700 blue. You take one of the bags and sample randomly with replacement after each chip. In 12 samples, you get 8 reds and 4 blues. What is the probability that this is urn $A$? Most subjects chose an answer around 0.7 whereas the correct answer according to Bayes’ Rule is closer to 0.97. Thus, they updated from 0.5 sluggishly as compared to the Bayesian update.

In our model, whether an agent exhibits conservatism or the base-rate fallacy is determined by the strength of the relevant association. To illustrate, consider the above urn experiment. There are two sources of uncertainty:
the identity of the urn (with possible realizations $\Omega_u = \{A, B\}$), and the proportion $n \in [0, 1]$ of red balls in 12 samples (with possible realizations $\Omega_p = \{0, \frac{1}{12}, ..., \frac{11}{12}, 1\}$). To ease notation, we denote a singleton menu by the element it contains.

For any $k > 0$, consider a linear increasing function $f_k(n) = k(n - \frac{1}{2}) + \frac{1}{2}$ that passes through $(0.5, 0.5)$. Suppose

$$\begin{align*}
r(B|n) &= k(n - \frac{1}{2}) + \frac{1}{2} \\
r(A|n) &= 1 - r(B|n).
\end{align*}$$

Therefore, the more red balls there are, higher the value of $r(B|n)$, implying a lower association with $B$ and in turn a higher association with $A$. Note also that higher the value of $k$, the more responsive the (reciprocal of the) association is to $n$. Let the prior association with the identity be symmetric:

$$r(A|\Omega_u) = r(B|\Omega_u).$$

Consider the value of $\frac{p(A|\frac{2}{3})}{p(B|\frac{2}{3})} = \frac{p(A, \Omega_p|\Omega_u, \frac{2}{3})}{p(B, \Omega_p|\Omega_u, \frac{2}{3})}$ at the proportion $n = \frac{2}{3}$ as in the experiment. The Bayesian posterior likelihood ratio is $\frac{0.97}{0.03}$.

**Proposition 2** There exists $k^* > 0$ s.t.

$$\frac{p(A|\frac{2}{3})}{p(B|\frac{2}{3})} \geq \frac{0.97}{0.03} \iff k \geq k^*.$$

If $k < k^*$ the agent is not responsive enough to the information and exhibits conservatism $\frac{p(A|\frac{2}{3})}{p(B|\frac{2}{3})} < \frac{0.97}{0.03}$, and if $k > k^*$ she is too responsive and exhibits base-rate neglect $\frac{p(A|\frac{2}{3})}{p(B|\frac{2}{3})} > \frac{0.97}{0.03}$.

\footnote{Here we are denoting the events (\{B\}, $\Omega_p$) and (\{A\}, $\Omega_p$) by $B$ and $A$ respectively, and the event (\$\Omega_u, \{n\}$) by $n$.}
This illustrates that, in our model, whether the agent exhibits conservatism or base-rate neglect depends on the strength of the associations given the data. In particular, both can coexist in our model.

5.0.2 Gambler’s Fallacy vs Hot-Hand Fallacy

KT point out that the Representativeness heuristic gives rise to a “law of small numbers”, whereby people expect population averages to obtain in small samples. Thus, for example, after 2 coins show heads, people overestimate the probability that the 3rd coin will come up tails, thinking that the composition of small samples must be similar to that of large ones. This particular example illustrates the so-called the gambler’s fallacy, also known as the Monte Carlo fallacy. The hot-hand fallacy goes the other way.

Let us interpret the outcome of each flip of a coin as a source of uncertainty. If heads on a toss is associated more strongly with a tails on others, then we obtain the gambler’s fallacy: if one heads and one tails have been realized then the agent considers a subsequent heads as likely as a subsequent tails, and in all other cases, the beliefs favor a mix of outcomes.

**Proposition 3** Suppose that the signal function satisfies

\[ a(t|h) > a(h|h), \ a(t|h) = a(h|t) \text{ and } a(t|t) = a(h|h) \ . \]

Then

\[ p(t|ht) = p(h|ht) \]

while

\[ p(t|hh) > p(h|hh) \text{ and } p(th|h) > p(hh|h). \]
The model constrains associative links to exist between pairs of elementary events only, and so in the preceding the outcome of the third toss cannot depend on the joint outcomes of the first two tosses in a nontrivial way. However, this modification can be obtained either by extending the model appropriately to allow for multi-node associations, or redefining sources of uncertainty such that an elementary event now describes a multi-dimensional event rather than a unidimensional one.

The hot-hand fallacy obtains in a similar way as the gambler’s fallacy by interpreting $h$ as making a shot and $t$ as failing to do so, and by assuming $a(t|h) < a(h|h)$.

6 Non-Additivity and Non-Monotonicity

We consider the evidence of non-additivity and non-monotonicity of beliefs in the psychology literature in this section. Our main model is flexible enough to accommodate these. Recall that Intuitive beliefs take the form

$$p(x|z) = \exp \left[ - \sum_{i,j \in \Gamma(x|z)} \left[ \pi(x_i|x_j) a(x_j|z) \right]^{-1} \right].$$

We show below that the model is flexible enough that violations of additivity are readily established. It is evident that Monotonicity in $x$ holds if both $\pi(x_i|x_j)$ and $a(x_j|z)$ are increasing in $x$ (wrt to set inclusion). It follows that if $\pi(x_i|x_j)$ is not increasing, then Monotonicity can be violated. These observations show that Intuitive beliefs model can accommodate the evidence discussed below.

However, in this section we introduce and focus on an interesting variant
of our model:

**Definition 8 (Non-Monotone Intuitive Beliefs)**  A Non-Monotone Intuitive Belief representation for $\pi$ is an associative network $(a, \pi)$ such that for any $x, z \in \Sigma$ s.t. $x \subset z$,

$$p(x|z) = \exp \left[ - \left( \sum_{i,j \in \Gamma(x,z)} \pi(x_i|x_j) a(x_j|z) \right)^{-1} \right],$$

where $x \subset x'$ implies $\pi(x_i|x_j) \geq \pi(x'_i|x'_j)$ and $a(x_j|z) \geq a(x'_j|z)$.

The difference lies in when the summation $\sum_{i,j \in \Gamma(x,z)}$ takes place. In the Intuitive beliefs model this takes places after the signals are inverted. In the Non-Monotone model, it takes places before the signals are inverted.

### 6.1 Evidence

#### 6.1.1 Non-Monotonicity: The Conjunction Fallacy

Consider the following example based on KT’s well known “Linda the bank teller” experiment. Let $f$ denote “feminist” and $b$ denote “bank teller”. There are two sources of uncertainty: Linda’s activism and her vocation. The possible realizations $\Omega_a$ of activism include $f$ and the possible realizations $\Omega_v$ of vocation include $b$. The agent is given some information, denoted $z = (z_a, z_v)$, about Linda’s possible activism and vocation. The conjunction fallacy is a violation of monotonicity of beliefs: it obtains when the belief $p(\{f\}, \{b\}|z_a, z_v)$ that Linda is a feminist bank teller is higher than the belief $p(\Omega_a, \{b\}|z_a, z_v)$ that she is a bank teller.

We show that
Proposition 4  The Non-Monotonic Intuitive Beliefs model exhibits: for all \( x, z \in \Sigma \) and \( l \in \Gamma \),

\[
p(x|z) \geq p(\Omega; x_{-l}|z).
\]

One way to think about the conjunction fallacy is that explicitly asking about a particular source of uncertainty can bring about awareness of that source, which then impacts beliefs. When the belief about Linda’s activism was not asked about, that source of uncertainty became irrelevant, and so the agent’s beliefs did not involve any consideration of activism. Indeed, the model will not produce a conjunction fallacy if activism remained a relevant state. For instance, if the agent is asked “Is Linda a feminist bank teller (\( \{ f\}, \{ b\} \)) or a bank teller who is either a feminist or an environmentalist (\( \{ f, e\}, \{ b\} \))”, and if the information \( z_a \) contains (say) \( f \), environmentalist and animal rights activist, then activism will be a relevant source and the agent will not exhibit the conjunction fallacy.

The model evidently always exhibits the conjunction fallacy, for any \( x, z \in \Sigma \) and \( l \in \Gamma \). But empirically one would not expect the fallacy to always exist. A comment relevant here is that the beliefs that are expressed empirically are not necessarily always intuitive, and may have a deliberative component as well. In such a case, consciously verifying that the laws of probability are not being violated would mitigate the conjunction fallacy. Our model on the other hand is aimed at modelling purely intuitive beliefs.

6.1.2 Non-Additivity: The Disjunction Fallacy

The disjunction fallacy arises when the sum of probabilities of events exceeds the probability of the union of events. For instance, it arises if the sum of an
agent’s belief that Linda is a bank teller and that she is a feminist exceeds the belief that Linda is either a bank teller or a feminist.

We find that Source Additivity (Section 4.4) can be violated in this manner:

**Proposition 5** *Non-Monotone Intuitive beliefs permit*

\[ p(x_i x_{-i}|z) + p(x'_i x_{-i}|z) \geq p(x_i \cup x'_i, x_{-i}|z) \]

*for some \( x, x', z \).*

To establish this, it suffices to show the desired inequality can hold for marginal beliefs:

\[ p(x_i|z) + p(x'_i|z) \geq p(x_i \cup x'_i|z). \]

In the model (and in fact also for Intuitive beliefs), this inequality translates to

\[ \exp[-r(x_i|z)] + \exp[-r(x'_i|z)] \geq \exp[-r(x_i \cup x'_i|z)], \]

which is easily satisfied with the appropriate choice of \( r. \)

## 7 Related Literature

### 7.1 Psychology

The classic Heuristics and Biases program of KT is based on the idea that people’s judgements are based on heuristics. These heuristics can be useful in many situations, but they can also give rise to systematic biases relative to the standard Bayesian model. KT organize the evidence they accumulated
under three heuristics: Representativeness, Availability, and Anchoring and Adjustment. We discuss the relationship between our model and each of these in turn:

The Anchoring and Adjustment heuristic describes the evidence that beliefs are responsive to possibly irrelevant anchors. This paper does not speak to that evidence.

The Availability heuristic organizes evidence that people judge as more likely those events for which they can readily recall more examples. KT emphasize salience as the determining feature. Salient examples and experiences may be responsible for stronger associations in our model, and in this sense, our model contains the spirit of the Availability heuristic. However, the notion of association is more general than that of salience: associations can be shaped by salience – for instance, a personal experience may create a stronger association than observing someone else’s experience – but they are also shaped by other factors such as the frequency of co-occurrence.

The Representativeness heuristic organizes evidence that people’s update reflects the representativeness of the information rather than the Bayesian update of prior beliefs. The heuristic dependence on representativeness of information gives rise to the updating biases discussed in Section 5. In formulating this heuristic, KT hypothesize that people’s updates mainly respond to similarity. In our model, updates respond to associations, and to the extent that similarity creates association, our model captures the spirit of Representativeness. However, as before, associations are a more general notion than similarity.

The use of the general notion of association rather than more specific
notions like salience and similarity has the advantage for economics that it can arguably be proxied by a calibration using empirical frequencies, thereby opening the door to empirical analysis.

### 7.2 Economics

Most research in economics generalizes Bayes’ Rule in order to accommodate particular findings. For instance, Rabin (2002) seeks to explain the law of small numbers. Epstein et al (2008) model updating that may be biased towards the prior or the data. Our model seeks to unify a range of findings.

Gennaioli and Shleifer (2010) seek to capture the Representativeness heuristic. Here is a very minimalistic outline of their model. When assessing the likelihood of some event $x \subset z$ given information $z$, the agent thinks only about the most “representative” or “stereotypical” subset $s(x) \subset x$, where representativeness is measured using the Bayesian conditional probability of $x$ given $s(x)$. Then the agent’s updated beliefs are given by

$$p^R(x|z) = \frac{p(s(x))}{p(s(x)) + p(s(x^c))},$$

that is, beliefs reflect the likelihood of the stereotypical event $s(x)$ reflective of $x$ relative to the stereotypical event $s(x^c)$ reflective of $x^c$. The model is shown to exhibit updating biases and the conjunction and disjunction fallacies. The model formalizes KT’s description of the Representativeness heuristic, while ours follows the latter less closely.

At least as relevant as the substantive literature on updating is the very narrow literature on belief formation. Spiegler (2016) hypothesizes that an agent’s model of the world may differ from the true model. A model here
is given by a DAG (directed acyclic graph) representing the presumed correlations between variables. Using her subjective model, the agent may use some conditional probabilities derived from data to formulate her joint beliefs, which will in general differ from the true joint distribution due to model misspecification. The agent’s beliefs are Bayesian with respect to her own model. To compare with our model: The associative network we use differs substantially from DAGs. Not only can it be undirected and cyclic, but the strength of links matters, whereas DAGs specify only the existence of links. Spiegler’s agent uses the standard Bayesian network formula to translate her data and DAG into a belief, whereas data is encoded in an unspecified manner into our agent’s associative network and the resulting beliefs will generically not satisfy the Bayesian network formula. Finally, associations do not have the same mathematical properties as probabilities. For instance, \( a(x|z) \) does not equal \( a(x|y)a(y|z) \).

The model of case-based inductive inference by Gilboa and Schmiedler (2003) considers an agent who ranks the likelihood of different eventualities in a given problem. The model relates memory banks of past cases with a likelihood relation on eventualities. This differs from the Bayesian model conceptually in that the case-based agent looks back at the past in order to evaluate likelihoods, whereas the Bayesian agent explicitly thinks in terms of her view of the world as it would exist in the future. To compare with our model: Other than being very different in spirit, our model retains the forward-looking nature of Bayesian beliefs but founds these beliefs on associations, which are backward-looking in the sense of being determined by past experiences. Moreover, while the memory bank is a primitive of the
case-based model, in our model the past experience is encoded in the agent’s subjective network. The case-based model also requires the modeler to specify a similarity function (see Argenziano and Gilboa 2018 for how similarity functions may be estimated from the data), while our model is based on associations which can arguably proxied by frequency data, as noted earlier. Finally, while the mechanical way in which beliefs are generated in our model makes it hard to interpret it in terms of deliberation, the case-based procedure can very readily be interpreted in terms of an agent generating beliefs through deliberation – without any other means to form beliefs about outcomes, she purposefully uses past data along with her assessment of similarity to help form likelihood assessments.

One implication of founding beliefs on associations that sharply distinguishes our model from the above two models is as follows: Associations can be built if the agent is faced with copies of the data. A news story that is aired continuously on television can form an association without one even being conscious of it, thereby affecting beliefs. But such repetition should not have any impact on the agents modelled in the above noted papers.

8 Conclusion

In order for an agent to arrive at a rational conclusion, she needs to engage in deliberation. However, the real world is sufficiently complex and cognitive/time resources sufficiently limited that deliberation will generically not be carried out to its conclusion. We hypothesize that after an agent has deliberated to the best of her abilities, she may let her intuition determine
her final choice. In this paper we articulate a way to conceptualize intuition and to model it mathematically in a manner that makes the model tractable and amenable to empirical application. We use Kahneman and Tversky’s empirical findings as minimal properties we would like our descriptive model to be disciplined by.

There are several directions for future work. A characterization of intuitive beliefs in terms of properties of beliefs would supply key tests for the model – this is pursued in Noor (2018). Another important direction is to theorize how an agent’s associative network may be shaped by the data. It may be, for instance, that the intuitive process can correctly learn the likelihood of simple events (e.g. marginal probabilities). One can theorize further. If an agent is already holding an asset, she may be more sensitive to parts of the data that speak to the possibility of making large gains or losses, and this may influence subsequent behavior. That is, the agent’s utility function may also enter a theory that links associative networks with data. With a theory of network formation in place, one can analyze how markets may react to changes in fundamentals.

We close with two comments on the role of intuition in rationality. The first is that if intuition is rooted in data, it should be viewed as a informative, albeit noisy, signal. Consequently, it should be considered rational to recommend an agent to follow their intuition after they have gone as far as their deliberation will take them. The second comment is based on the observation that deliberation is little more than the “cleaning up” of intuition. That is, deliberation is an operation on the material supplied by intuition. If deliberative conclusions are constructed from intuition, then properties
of intuition may restrict where an agent’s deliberation will take them. In particular, deliberation may not guarantee the attainment of rationality.

A Appendix

A.1 Regularity

Say that an Intuitive belief is regular if it satisfies the following uncontentious properties:

R1. If $x \subseteq x'$ and $z \subseteq z'$, then $[p(x|z) > 0 \implies p(x'|z') > 0]$ and $[p(x'|z) < 1 \implies p(x|z') < 1]$.

R2. If $x \subseteq z$, then $[p(x|\Omega) > 0 \implies p(x|z) > 0]$.

R3. For any nonsingleton $x_i$, $[p(x|z) > 0 \iff \exists y_i \subseteq x_i \text{ s.t. } p(y;x_i|z) > 0]$.

R4. For any $x_i, y_i$, $[p(x_i|z) = 0 \implies p(x_i \cup y_i|z) = p(y_i|z)]$.

R5. If $x_i \subseteq z_i$, $[p(x_i|\Omega,z_i) = 1 \implies p(x_i|z_i x_i) = 1]$.

To paraphrase, R1 states that if $x$ is possible given $z$, then larger events are possible under less information, and moreover, if $x'$ is uncertain given $z$, then smaller events are uncertain under less information. For an example of the first part of R1: if it is possible that it will rain when it is very cloudy, then it is possible that there will be precipitation if it is cloudy. R2 states that if $z$ and $x \subseteq z$ are possible ex ante, then $x$ is possible also conditional on information $z$. For instance, if it is possible that it will rain, then it is possible that it will rain conditional on precipitation. R3 states that a non-singleton event is possible if and only if contains a possible event. If we take
to be countable, then singleton events can be strictly positive, and in this case R3 implies that an event is possible if and only if it contains a possible singleton event. R4 says that impossible events play no role in determining the belief about any event. R5 says that if \( x_i \) is certain ex ante, then it is certain also conditional on any information that does not rule it out.

We first show that structured Intuitive beliefs are regular.

**Lemma 1** For any structured Intuitive belief \( p \), the following properties hold

(i) (Monotonicity) \( p \) satisfies \( x \subset x' \implies p(x|z) \leq p(x'|z) \) and \( x \subset z' \implies p(x|z) \geq p(x|z') \)

(ii) \( p \) is regular.

**Proof.** Proof of Monotonicity: By the functional forms, \( x \subset x' \) implies

\[
\tau(x_i|x_j) \geq \tau(x'_i|x'_j)
\]

for all \( i \neq j \) and \( r(x_j|z) \geq r(x'_j|z) \) for all \( j \), since

\[
\tau(x_i|x_j) = \left[ \sum_{x_i \in x_i} \sum_{x_j \in x_j} \alpha(x_i|x_j) \right]^{-1} \geq \left[ \sum_{x_i \in x'_i} \sum_{x_j \in x'_j} \alpha(x_i|x_j) \right]^{-1} = \tau(x'_i|x'_j)
\]

for all \( i \neq j \)

and \( r(x_j|z) := \sum_{k \in \Gamma} \sum_{x_j \in x_j} \sum_{z_k \in z_k} a(x_j|z_k) \geq \sum_{k \in \Gamma} \sum_{x_j \in x'_j} \sum_{z_k \in z_k} a(x_j|z_k) = r(x'_j|z) \)

Consequently, \( p(x|z) \leq p(x'|z') \).

Similarly, if \( z \subset z' \) then \( r(x_j|z) \leq r(x_j|z') \) since

\[
r(x_j|z) := \sum_{k \in \Gamma} \sum_{x_j \in x_j \setminus x_j} \sum_{z_k \in z_k} a(x_j|z_k) \leq \sum_{k \in \Gamma} \sum_{x_j \in x'_j \setminus x_j} \sum_{z_k \in z_k} a(x_j|z_k) = r(x'_j|z')
\]

Consequently, \( p(x|z) \geq p(x|z') \).

Proof of R1: Follows directly from the model’s monotonicity property wrt \( x \) and \( z \).

Proof of R2: Suppose \( x \subset z \) and \( p(x|\Omega) > 0 \). By the representation, it is clear that \( \tau(x_i|x_j) < \infty \) for each \( i, j \). We show next that \( r(x_j|z) < \infty \) for
each \( j \), as a result of which, given the representation, we obtain the desired result that \( p(x|z) > 0 \).

By R1, \( p(x_j|\Omega) > 0 \) for each \( j \in \Gamma \). But then \( r(x_j|\Omega) = \sum_{k \in \Gamma} \sum_{x_j \in \Omega_j} \sum_{x_k \in \Omega_k} a(x_j|z_k) < \infty \). This implies that \( a(x_j|z_k) < \infty \) for each \( x_j \in \Omega_j \setminus x_j \), \( k \in \Gamma \) and \( z_k \in \Omega_k \). Consequently \( a(x_j|z_k) < \infty \) for each \( x_j \in \mathbf{z}_j \setminus x_j \), \( k \in \Gamma \) and \( z_k \in \mathbf{z}_j \) and it follows that \( r(x_j|z) := \sum_{k \in \Gamma} \sum_{x_j \in \mathbf{z}_j \setminus x_j} \sum_{x_k \in \mathbf{z}_k} a(x_j|z_k) < \infty \), as desired.

Proof of R3: If \( p(x|z) > 0 \) then it is necessary that \( \pi(x_i|x_j), \pi(x_j|x_i), r(x_i|z) < \infty \). If \( x_i \) is not a singleton, then there is some \( x_i, x'_i, x''_i \in x_i, x_j, x'_j \in x_j, z'_i \in \mathbf{z}_i \setminus x_i \) and \( z \in \mathbf{z} \) s.t. \( \pi(x_i|x_j) > 0, \pi(x'_j|x'_i) > 0 \) and \( \sum_{k \in \Gamma} a(z'_i|z_k) > 0 \).

Let \( y_i = \{x_i, x'_i\} \). Then \( \pi(y_i|x_j), \pi(x_j|y_i), \pi(y_i|z) < \infty \) and all other terms are unaffected. So we must have \( p(y_i;\mathbf{x}_{-i}|z) > 0 \).

Proof of R4: Suppose \( p(x_i|z) = 0 \). By the functional forms,

\[
p(x_i|z) = 0 \\
\implies r(x_i|z) = \infty \\
\implies a(x_i|z_k) = 0 \text{ for all } x_i \in \mathbf{z}_i \setminus x_i, z_k \in \mathbf{z}_k \text{ and } k.
\]

But then

\[
p(x_i \cup y_i|z) \\
= \exp\left[-\sum_{x_j \in \mathbf{z}_j \setminus x_j, y_j} \sum_{k \in \Gamma} \sum_{x_k \in \mathbf{z}_k} a(x_j|z_k)\right] \\
= \exp\left[-\sum_{x_j \in \mathbf{z}_j \setminus x_j} \sum_{k \in \Gamma} \sum_{x_k \in \mathbf{z}_k} a(x_j|z_k) + \sum_{x_j \in \mathbf{z}_j \setminus y_j} \sum_{k \in \Gamma} \sum_{x_k \in \mathbf{z}_k} a(x_j|z_k)\right] \\
= \exp\left[-\sum_{x_j \in \mathbf{z}_j \setminus y_j} \sum_{k \in \Gamma} \sum_{x_k \in \mathbf{z}_k} a(x_j|z_k)\right] \\
= p(y_i|z), \text{ as desired.}
\]

Proof of R5: Follows from the model’s monotonicity property wrt \( z \). ■
A.2 Proof of Theorem 1

We prove a more general result that requires beliefs to be regular rather than structured. Then Theorem 1 is obtained as a corollary.

Consider a regular Intuitive belief $p$.

Lemma 2 For any $x, z$ s.t. $p(z|\Omega) > 0$,

$$p(x|z) = \prod_{j \in \Gamma(x|z)} \sum_{i \in \Gamma(x|z)} \tau(x_i|x_j) \cdot \pi(x_i|x_j).$$

Proof. We see that beliefs can be written as:

$$p(x|z) = \exp \left[ - \sum_{j \in \Gamma(x|z)} \sum_{i \in \Gamma(x|z)} \tau(x_i|x_j) r(x_j|z) \right]$$

$$= \prod_{j \in \Gamma(x|z)} \exp \left[ - \sum_{i \in \Gamma(x|z)} \tau(x_i|x_j) r(x_j|z) \right]$$

$$= \prod_{j \in \Gamma(x|z)} \exp \left[ - r(x_j|z) \left( \sum_{i \in \Gamma(x|z)} \tau(x_i|x_j) \right) \right]$$

$$= \prod_{j \in \Gamma(x|z)} p(x_j|z)^{\sum_{i \in \Gamma(x|z)} \tau(x_i|x_j)} \text{ since } p(x_i|z) = \exp[-r(x_i|z)]$$

$$= \prod_{j \in \Gamma(x|z)} p(x_j|z)^{\sum_{i \in \Gamma(x|z)} \tau(x_i|x_j)} \text{ since } j \not\in \Gamma(x|z) \text{ implies } p(x_j|z) = 1,$$

as desired. ■

As a corollary, we note in particular that, since $\tau(x_i|x_i) = 1$, we can write

Lemma 3 For any $x, x_j, z$,

$$p(x, x_j|z) = p(x_i|z)^{1+\tau(x_j|x_i)} p(x_j|z)^{1+\tau(x_i|x_j)}$$

We use this expression below freely without reference to the lemma.
Lemma 4  For any $x, z$ s.t. $p(z|\Omega) > 0$, $[x_i \neq \Omega_i \implies x_i \not\subseteq z_i]$ and $p(x|z) > 0$,

$$\frac{p(x|z_i z_{-i})}{p(x|\Omega_i z_{-i})} = \frac{p(x|z_i \Omega_{-i})}{p(x|\Omega_i \Omega_{-i})}.$$ 

Proof. We first show that for any $x, z$ s.t. $[x_i \neq \Omega_i \implies x_i \not\subseteq z_i]$,

\[
\Gamma(x|z) = \Gamma(x|\Omega_i z_{-i}).
\]

In general it is the case that $\Gamma(x|z) \subset \Gamma(x|\Omega_i z_{-i})$, $i \in \Gamma(x|z)$

$\implies p(x_i|z) < 1$

$\implies p(x_i|\Omega_i z_{-i}) < 1$ by R1

$\implies i \in \Gamma(x|\Omega_i z_{-i})$

However, by definition,

$i \in \Gamma(x|\Omega_i z_{-i})$

$\implies p(x_i|\Omega_i z_{-i}) < 1$

$\implies x_i \neq \Omega_i$

$\implies x_i \not\subseteq z_i$ by hypothesis

$\implies p(x_i|z_i z_{-i}) < 1$ by R5

$\implies i \in \Gamma(x|z)$.

Consequently, $\Gamma(x|z) = \Gamma(x|\Omega_i z_{-i})$, as desired.

Now to prove the desired claim, take any $x, z$ s.t. $[x_i \neq \Omega_i \implies x_i \not\subseteq z_i]$ and $p(x|z) > 0$. If $\Gamma(x|z) = \phi$ then by the preceding, $\Gamma(x|z_i z_{-i}) = \Gamma(x|\Omega_i z_{-i}) = \Gamma(x|z_i \Omega_{-i}) = \phi$, and so $p(x_j|z_i z_{-i}) = p(x_j|\Omega_i z_{-i}) = p(x_j|z_i \Omega_{-i}) = p(x_j|\Omega_i \Omega_{-i}) = 1$. Then $\frac{p(x|z_i z_{-i})}{p(x|\Omega_i z_{-i})} = \frac{p(x|z_i \Omega_{-i})}{p(x|\Omega_i \Omega_{-i})}$ holds trivially. Next, suppose $\Gamma(x|z) \neq \phi$. To ease notation, write $\Gamma(x|z) = \Gamma(x|\Omega_i z_{-i}) = I$. By R1,
Lemma 5

Suppose \( p \) is an Intuitive Belief that satisfies R1 and R5. Then for any \( \mathbf{x}, \mathbf{x}_j \) the following are equivalent:

(i) there exists a representation \((a, \pi)\) where \( \pi(\mathbf{x}_i | \mathbf{x}_j) = \pi(\mathbf{x}_j | \mathbf{x}_i) = 0 \) for all \( \mathbf{x}_i, \mathbf{x}_j \) s.t. \( p(\mathbf{x}_i, \mathbf{x}_j | \Omega) > 0 \).

(ii) \( p(\mathbf{x}_i, \mathbf{x}_j | \mathbf{z}) = p(\mathbf{x}_i | \mathbf{z}) p(\mathbf{x}_j | \mathbf{z}) \) for all \( \mathbf{z} \) s.t. \( p(\mathbf{z} | \Omega) > 0 \) and \( p(\mathbf{x}_i, \mathbf{x}_j | \mathbf{z}) > 0 \).

(iii) \( p(\mathbf{x}_i, \mathbf{x}_j | \Omega) = p(\mathbf{x}_i | \Omega) p(\mathbf{x}_j | \Omega) \) whenever \( p(\mathbf{x}_i, \mathbf{x}_j | \Omega) > 0 \).
Proof. We begin by making an observation that we rely on below: For any \( x_i, x_j \) and any \( z \) s.t. \( \Gamma(x_i, x_j | z) = \{i, j\} \) and \( p(x_i, x_j | z) > 0 \),

\[
p(x_i, x_j | z) = p(x_i | z)p(x_j | z)
\]

\( \iff \) \( \mathbb{I}(x_i | x_j) r(x_j | z) + \mathbb{I}(x_j | x_i) r(x_i | z) = 0. \) (1)

That is, beliefs about events with only two relevant sources can be written as a product of the marginals if and only if the indirect signals sum to zero.

This holds because

\[
p(x_i, x_j | z) = p(x_i | z)p(x_j | z)
\]

\( \iff \) \( \exp[-[(1 + \mathbb{I}(x_i | x_j)) r(x_j | z) + (1 + \mathbb{I}(x_j | x_i)) r(x_i | z)]] = \exp[-r(x_i | z) + r(x_j | z)]
\]

\( \iff \) \( \mathbb{I}(x_i | x_j) r(x_j | z) + \mathbb{I}(x_j | x_i) r(x_i | z) = 0. \)

Now we prove the lemma.

proof of (i) \( \iff \) (ii) : For sufficiency, take any \( x_i, x_j, z \) s.t. \( p(z | \Omega) > 0 \) and \( p(x_i, x_j | z) > 0 \). WLOG we can take \( x_i \subseteq z_i, x_j \subseteq z_j \). By R1, \( p(x_i, x_j | \Omega) > 0 \), and so we can invoke the hypothesis (that is, \( \mathbb{I}(x_i | x_j) = \mathbb{I}(x_j | x_i) = 0 \)) and see that (1) is satisfied. Indeed, if \( \Gamma(x_i, x_j | z) = \{i, j\} \), this implies \( p(x_i, x_j | z) = p(x_i | z)p(x_j | z) \), as desired. On the other hand, if \( \Gamma(x_i, x_j | z) = \phi \), then we have \( p(x_i | z) = p(x_j | z) = 1 \) by definition, and \( p(x_i, x_j | z) = p(x_i | z)^{1+\mathbb{I}(x_i | x_j)} p(x_j | z)^{1+\mathbb{I}(x_j | x_i)} = 1 = p(x_i | z)p(x_j | z) \), as desired. Finally, consider the case where \( \Gamma(x_i, x_j | z) \neq \{i, j\} \), and suppose WLOG that \( \Gamma(x_i, x_j | z) = \{j\} \). Then \( p(x_i | z) = 1 \) and moreover, \( p(x_i, x_j | z) = p(x_i | z)^{1+\mathbb{I}(x_i | x_j)} p(x_j | z)^{1+\mathbb{I}(x_j | x_i)} = p(x_j | z) \) since \( p(x_i | z) = 1 \) and \( \mathbb{I}(x_i | x_j) = 0 \). Therefore again, \( p(x_i, x_j | z) = p(x_i | z)p(x_j | z) \), as desired. This establishes sufficiency.
For necessity, take any \(x, x_j\) s.t. \(p(x, x_j|\Omega) > 0\). If \(i \notin \Gamma(x, x_j|z')\) or \(j \notin \Gamma(x, x_j|z')\) for all \(z'\) then the terms \(\pi(x_i|x_j)\) and \(\pi(x_j|x_i)\) never appear in the representation (because the representation sums terms only over relevant sources). Consequently WLOG we can set \(\pi(x_i|x_j) = \pi(x_j|x_i) = 0\). Suppose next that \(\Gamma(x, x_j|z) = \{i, j\}\) for some \(z\), in which case (1) holds. By R1 we can let \(z = \Omega\). If \(r(x_j|\Omega) \neq 0\), then given that \(r, \pi \geq 0\), the equality (1) implies that \(\pi(x_i|x_j) = 0\). However, if \(r(x_j|\Omega) = 0\), then by definition, \(p(x_j|\Omega) = 1\) and for any \(z\) s.t. \(x_j \subset z\), R5 implies \(p(x_j|z) = 1\) and \(r(x_j|z) = 0\). Consequently \(\pi(x_i|x_j)\) is always multiplied by 0 in any expression where \(\pi(x_i|x_j)\) appears, and so WLOG we can take \(\pi(x_i|x_j) = 0\). By an analogous argument, \(\pi(x_j|x_i) = 0\). This completes the proof for necessity.

proof of (i) \(\iff\) (iii) : Sufficiency follows trivially from (i) \(\implies\) (ii) \(\implies\) (iii). For necessity, argue as in the proof of [(ii) \(\implies\) (i)].

Observe that condition (ii) is weaker than Positive Statistical Independence since it applies only to events of which at most two sources are relevant. Nevertheless:

**Lemma 6** Suppose \(p\) is an Intuitive Belief that satisfies R1 and R5. Then \(p\) satisfies Positive Statistical Independence iff it admits a representation \((a, \pi)\) with \(\pi(x_i|x_j) = 0\) for all \(x_i, x_j\) s.t. \(p(x, x_j|\Omega) > 0\).

**Proof.** Sufficiency follows trivially from Lemma 5, as Positive Statistical Independence implies condition (ii) stated there, which is shown to be equivalent to condition (i) stated there, as desired.

For necessity first note that for any \(x, z\) s.t. \(p(x|z) > 0\), R1 implies \(p(x, x_j|z) > 0\) and in turn \(p(x, x_j|\Omega) > 0\) for any \(i, j\). Therefore by hypothesis, \(\pi(x_i|x_j) = \pi(x_j|x_i) = 0\). We use this to establish that
\[ p(x|z) = \exp \left[ -\sum_{i \in \Gamma(x|z)} \sum_{j \in \Gamma(x|z)} r(x_i|x_j) r(x_j|z) \right] \]
\[ = \exp \left[ -\sum_{j \in \Gamma(x|z)} r(x_j|z) \right] \]
\[ = \prod_{j \in \Gamma(x|z)} \exp \left[ -r(x_j|z) \right] \]
\[ = \prod_{j \in \Gamma(x|z)} p(x_j|z) \]
\[ = \prod_{j \in \Gamma(x|z)} p(x_j|z), \]

where the last equality follows since \( j \not\in \Gamma(x|z) \implies p(x_j|z) = 1 \). This completes the proof. \( \blacksquare \)

Now we prove the following theorem. Theorem 1 is as a corollary, since structured beliefs are regular by Lemma 1.

**Theorem 3** Suppose \( p \) is a regular Intuitive Belief that satisfies richness. Then \( p \) is Bayesian if and only if it satisfies

(i) (Positive Statistical Independence): for all \( x, z \in \Sigma \) s.t. \( p(x|z) > 0 \),

\[ p(x|z) = \prod_{i \in \Gamma} p(x_i|z). \]

(ii) (Marginal Conditioning) for all \( x_i, z \) s.t. \( p(x_i|z) > 0 \),

\[ p(x_i|z) = \frac{p(x_i \cap z|\Omega)}{p(z|\Omega)}. \]

**Proof.** For necessity, take any \( x, z \) s.t. \( p(z|\Omega) > 0 \). If \( p(x \cap z|\Omega) = 0 \) then by R1 \( p(x|z) = p(x \cap z|z) = 0 \). Consequently Bayesian Conditioning is satisfied as \( p(x|z) = 0 = \frac{p(x \cap z|\Omega)}{p(z|\Omega)} \). Next consider the case where \( p(x \cap z|\Omega) > 0 \). Then by R2, \( p(x|z) = p(x \cap z|z) > 0 \), and so:

\[ p(x|z) = \prod_{i \in \Gamma} p(x_i|z) \text{ by Positive Statistical Independence} \]
\[ = \prod_{i \in \Gamma} \frac{p(x_i \cap z|\Omega)}{p(x_i|z)} \text{ since } p(x_i|z) = \frac{p(x_i|\Omega)}{p(z|\Omega)} \text{ by hypothesis} \]
= \prod_{i \in \Gamma} p(x_i \cap z_i | \Omega) \\
\prod_{i \in \Gamma} p(z_i | \Omega) \\
= \frac{p(x_i \cap z_{\Gamma} | \Omega)}{p(z_{\Gamma} | \Omega)},
establishing that beliefs are Bayesian.

We prove sufficiency in steps. The first step establishes a result for events that are not singletons, and step 2 extends this to all events.

**Step 1:** Show that for any \( z_i z_j \) s.t. \( p(z_i z_j | \Omega) > 0 \), if there exists \( x_i \nsubseteq z_i \) s.t. \( p(x_i | z_i z_j) > 0 \) then

\[ p(z_i z_j | \Omega) = p(z_i | \Omega) p(z_j | \Omega). \]

Take any \( z_i z_j \) s.t. \( p(z_i z_j | \Omega) > 0 \) and suppose there is \( x_i \nsubseteq z_i \) s.t. \( p(x_i | z_i z_j) > 0 \). Then \( x = x_i \) satisfies the conditions in Lemma 4, and the expression derived there yields:

\[ p(x_i | z_i z_j) = \frac{p(x_i | z_i)}{p(x_i | \Omega)} p(x_i | z_j). \]

Since \( x_i \nsubseteq z_i \), we see

\[ \frac{p(x_i z_j | \Omega)}{p(z_i z_j | \Omega)} = p(x_i | z_i z_j) \quad \text{(by Bayesian conditioning)} \]

\[ = \frac{p(x_i | z_i)}{p(x_i | \Omega)} p(x_i | z_j) \quad \text{(by Lemma 4)} \]

\[ = \frac{p(x_i | \Omega)}{p(z_i | \Omega)} p(x_i z_j | \Omega) \quad \text{(by Bayesian conditioning)} \]

\[ = \frac{p(x_i z_j | \Omega)}{p(z_i | \Omega) p(z_j | \Omega)}, \]

that is, \( \frac{p(x_i z_j | \Omega)}{p(z_i z_j | \Omega)} = \frac{p(x_i z_j | \Omega)}{p(z_i | \Omega) p(z_j | \Omega)} \) and the desired assertion follows.

**Step 2:** Show that for any \( x_i x_j \) s.t. \( p(x_i x_j | \Omega) > 0 \),

\[ p(x_i x_j | \Omega) = p(x_i | \Omega) p(x_j | \Omega). \]
Take any \( x, x_j \) s.t. \( p(x, x_j | \Omega) > 0 \). By R1, \( p(x_i | \Omega) > 0 \).

First suppose that \( x_i \) is not a singleton, then by R3 there exists \( y_i \not\subseteq x_i \) s.t. \( p(y_i, x_j | x, x_j) > 0 \) in which case it is also true by definition of a belief that \( p(y_i | x, x_j) = p(y_i | x_j) > 0 \). Consequently, we are in the case of Step 1 and the assertion follows.

Now suppose \( x_i \) is a singleton, \( x_i = \{ x_i \} \). As before, \( p(x_i | \Omega) > 0 \). By Richness, there exists \( x_i' \in \Omega_i \) that is distinct from \( x_i \) and it must be that \( p(\{ x_i, x_i' \} | \Omega) < 1 \). By R3, \( p(\{ x_i, x_i' \} | \Omega) > 0 \) as well. Write \( z_i = \{ x_i, x_i' \} \). Since \( 0 < p(z_i | \Omega) < 1 \), it must be that \( z_i \not\subseteq \Omega_i \). The event \( z_i x_j \) then satisfies the conditions of Step 3 by construction and therefore,

\[
p(z_i x_j | \Omega) = p(z_i | \Omega)p(x_j | \Omega).
\]

Next, observe that

\[
\frac{p(x, x_j | \Omega)}{p(z_i | \Omega)} = \frac{p(x, x_j | z_i)}{p(z_i | \Omega)} \text{ (by Bayesian conditioning)}
\]

\[
= \frac{p(x_j | z_i)^{1+\tau(x_j | x_i)}p(x_j | z_i)^{1+\tau(x_j | x_i)}}{p(z_i | \Omega)^{1+\tau(x_i | x_j)}}
\]

\[
= \frac{p(x_j | \Omega)^{1+\tau(x_j | x_i)}p(z_i | \Omega)^{1+\tau(x_i | x_j)}}{p(z_i | \Omega)^{1+\tau(x_i | x_j)}} \text{ (by (2))}
\]

\[
= \frac{p(x_j | \Omega)^{1+\tau(x_j | x_i)}p(x_j | \Omega)^{1+\tau(x_j | x_i)}}{p(z_i | \Omega)^{1+\tau(x_j | x_i)}}
\]

\[
= \frac{p(x_j | \Omega)^{1+\tau(x_j | x_i)}}{p(z_i | \Omega)^{1+\tau(x_j | x_i)}}.
\]

That is, \( \frac{p(x_i x_j | \Omega)}{p(z_i | \Omega)} = \frac{p(x_j | \Omega)}{p(z_i | \Omega)^{1+\tau(x_j | x_i)}} \), which implies

\[
p(z_i | \Omega)^{\tau(x_j | x_i)} = 1.
\]
Since \( p(z_i|\Omega) < 1 \) by construction, it must be that \( \tau(x_j|x_i) = 0 \). A symmetric argument yields \( \tau(x_i|x_j) = 0 \), and the result then follows from Lemma 5.

**Step 3:** Show that for any \( x, z \) s.t. \( p(z|\Omega) > 0 \) and \( p(x|z) > 0 \),

\[
p(x|z) = \prod_{i \in \Gamma} p(x_i|z).
\]

By Lemma 5, the result in Step 2 implies that \( p \) admits a representation \( (\alpha, \pi) \) with \( \tau(x_i|x_j) = 0 \) for all \( x_i, x_j \) s.t. \( p(x_i,x_j|\Omega) > 0 \). By Lemma 6, Positive Statistical Independence is then implied.

**Step 4:** Show that for any \( x, z \) s.t. \( p(z|\Omega) > 0 \) and \( p(x|z) > 0 \),

\[
p(x_j|z) = \frac{p(x_j \cap z_j|\Omega)}{p(z_j|\Omega)} \text{ for all } j.
\]

Consider any \( x, z \) s.t. \( p(z|\Omega) > 0 \) and \( p(x|z) > 0 \). Since \( p(x|z) > 0 \), R1 implies that for any \( j \), \( p(x_j|z) > 0 \). By definition of a belief, \( p(x \cap z|z) = p(x|z) > 0 \), and then R1 also implies that \( p((x_j \cap z_j)z_{-j}|\Omega) > 0 \) for any \( j \), since \( x \cap z \subset (x_j \cap z_j)z_{-j} \). Therefore by Step 2, and the fact that beliefs are Bayesian we obtain

\[
p(x_j|z) = \frac{p((x_j \cap z_j)z_{-j}|\Omega)}{p(z_j|\Omega)} \prod_{j \neq k \in \Gamma} \frac{p(z_k|\Omega)}{p(z_j|\Omega)} \prod_{j \neq k \in \Gamma} \frac{p(z_k|\Omega)}{p(z_j|\Omega)}.
\]

This completes the proof. \( \blacksquare \)

### A.3 Proof of Theorem 2

To prove necessity, suppose \( p \) satisfies Marginal Additivity and Positive Statistical Independence, and take any \( x, x', z \in \Sigma \) and \( i \in \Gamma \) s.t. \( x_i \cap x'_i = \phi \). If \( p(x_i \cup x'_i, x_{-i}|z) = 0 \), then by R1, \( p(x_i|x_{-i}|z) = p(x'_i|x_{-i}|z) = 0 \) and so
for any Independence we show that there exists a representation where Marginal Additivity is implied trivially. To show that Positive Statistical

by R1, we show that the term takes the form

where the last equality relies on the fact that by R1, \( p(x_i \cup x'_i, x_{-i} | z) > 0 \) implies \( p(x_i, x_{-i} | z) > 0 \) and \( p(x'_i, x_{-i} | z) > 0 \) and so these beliefs equal the product of their marginals (by Positive Statistical Independence). The completes the proof for necessity.

To prove sufficiency, assume Source Additivity is satisfied. First observe that Marginal Additivity is implied trivially. To show that Positive Statistical Independence we show that there exists a representation where \( \pi(x_i | x_j) = 0 \) for all \( x_i, x_j \) s.t. \( p(x_i, x_j | \Omega) > 0 \). By Lemma 6, Positive Statistical Independence is then implied.

So take any \( x_i, x_j \) s.t. \( p(x_i, x_j | \Omega) > 0 \). By R2, \( p(x_i | z) > 0 \) and \( p(x_j | z) > 0 \), for any \( z \) s.t. \( x_i \subset z_i \) and \( x_j \subset z_j \). Consider the following cases.

Case (a): \( p(x_i | z) = 1 \) for all \( z \) s.t. \( x_i \subset z_i \) and \( x_j \subset z_j \)

Then \( a(x_i | z) = \infty \) and indeed \( i \not\in \Gamma(x_i, x_j, x_{-ij} | z) \) regardless of \( x_{-ij} \). Therefore the term \( \pi(x_j | x_i) \pi(x_i | z) \) never appears in the representation.\(^{13}\) Consequently, we can let \( \pi(x_j | x_i) = 0 \) WLOG.

Case (b): \( p(x_i | z) < 1 \) for some \( z \) s.t. \( x_i \subset z_i \) and \( x_j \subset z_j \) but \( p(x_j | z) = 1 \) for all such \( z \).

\(^{13}\)Note that \( \pi(x_j | x_i) \) can appear only in \( p(x_i, x_j, x_{-ij} | z) \) where \( x_i \subset z_i \) and \( x_j \subset z_j \): indeed if \( x_i \not\subset z_i \) or \( x_j \not\subset z_j \) then term takes the form \( \pi(x_j \cap z_j | x_i \cap z_i) \) where \( x_i \cap z_i \neq x_i \) or \( x_j \cap z_j \neq x_j \), whereas we are specifically seeking the value of \( \pi(x_j | x_i) \).
Similar to the previous case, it must be that whenever \( x_i \subset z_i \) and \( x_j \subset z_j \), it is the case that \( j \notin \Gamma(x_i, x_{-ij}|z) \) regardless of \( x_{-ij} \). Consequently the term \( \tau(x_j|x_i) \) never appears in the expression for any \( p(x_i, x_{-ij}|z) \).

Case (c): \( 0 < p(x_i|z) < 1 \) and \( 0 < p(x_j|z) < 1 \) for some \( z \) s.t. \( x_i \subset z_i \) and \( x_j \subset z_j \).

Then \( \Gamma(x_i, x_j|z) = \{i, j\} \) and

\[
p(x_i, x_j|z) = \exp\left[-[1 + \tau(x_i|x_j))r(x_j|z) + (1 + \tau(x_j|x_i))r(x_i|z)]\right]
= p(x_j|z) (1 + \tau(x_i|x_j)) p(x_i|z) (1 + \tau(x_j|x_i)).
\]

Since \( p(x_i, x_j|\Omega) > 0 \) by hypothesis, R2 implies that \( p(x_i, x_j|z) > 0 \), which in turn implies that \( \tau(x_i|x_j) < \infty \) and \( \tau(x_j|x_i) < \infty \).

Choose \( x'_j \) s.t. \( x_j \cap x'_j = \emptyset \) and \( x_j \cup x'_j = z_j \). Since \( 0 < p(x_j|z) < 1 \), it must be that \( 0 < p(x'_j|z) < 1 \) as well, since Source Additivity requires \( p(x_j|z) + p(x'_j|z) = p(z_j|z) = p(\Omega_j|z) = 1 \). Observe also that by Source Additivity,

\[
p(x_i|z) = p(x_i, z_{-i}|z) \quad \text{(by definition of a belief)}
= p(x_i, x_j, z_{-ij}|z)
= p(x_i, x_j \cup x'_j, z_{-ij}|z)
= p(x_i, x_j \cup x'_j|z)
= p(x_i, x_j|z) + p(x_i, x'_j|z)
= p(x_j|z) (1 + \tau(x_i|x_j)) p(x_i|z) (1 + \tau(x_j|x_i)) + p(x'_j|z) (1 + \tau(x_i|x'_j)) p(x_i|z) (1 + \tau(x'_j|x_i))
\]

that is,

\[
p(x_j|z) (1 + \tau(x_i|x_j)) p(x_i|z) (1 + \tau(x_j|x_i)) = 1.
\]
Therefore by letting $\alpha = p(x_j|z)^{(1+\pi(x_i|x_j))} > 0$ and $\beta = p(x'_j|z)^{(1+\pi(x_i|x'_j))} \geq 0$ we can write

$$\alpha p(x_i|z)^{\pi(x_i|x_i)} + \beta p(x_i|z)^{\pi(x'_i|x_i)} = 1.$$ 

Since $0 < p(x_j|z) < 1$ and $\pi(x_i|x_j) < \infty$, it must be that $0 < \alpha < 1$. We show that it is possible to vary $z$ in a manner that does not change $\alpha, \beta$ but changes $p(x_i|z)$ to some different value that is also strictly between 0 and 1 (as a result of which the above equation is still valid). It then follows that $\pi(x_j|x_i) = 0$.

Since $p(x_i|z) < 1$, it must be that $x_i \not\subseteq z_i$ and in particular, there exists some $x_i \in z_i \setminus x_i$ s.t. $p(\{x_i\}|z) > 0$ (if $p(\{x_i\}|z) = 0$ for all such $x_i$ then by Source Additivity $p(z_i \setminus x_i|z) = 0$ and by R4 and the definition of beliefs, $p(x_i|z) = p(x_i \cup (z_i \setminus x_i)|z) = p(z_i|z) = 1$, a contradiction). Take any $k \neq i, j$, and first note that whether we increase or decrease $z_k$ to $z'_k$ (wrt set inclusion) it must be that $p(\{x_i\}|z'_k z_{-k}) > 0$: if we increase it then it is implied by R1. If we decrease it, then because richness* implies that $a(x'_i|z_j) > 0$ for any $z_j \in z_j$ and any $x'_i \in z_i \setminus \{x_i\}$, it must be that $p(\{x_i\}|z'_k z_{-k}) > 0$. Next note that we cannot have $p(x_i|z'_k z_{-k}) = 1$, since Source Additivity and $p(\{x_i\}|z'_k z_{-k}) > 0$ would imply that $p(x_i \cup \{x_i\}|z'_k z_{-k}) = p(x_i|z'_k z_{-k}) + p(\{x_i\}|z'_k z_{-k}) > p(x_i|z'_k z_{-k}) = 1$, a contradiction. Therefore, we can increase or decrease $z_k$ to $z'_k$ and we would have $p(x_i|z'_k z_{-k}) < 1$. As before, richness* implies that $a(x_i|z_j) > 0$ for any $z_j \in z_j$. Therefore, we also have $p(x_i|z'_k z_{-k}) > 0$. Moreover, again by richness*, in fact $p(x_i|z) \neq p(x_i|z'_k z_{-k})$.

The following Lemma confirms that such a change would not change $\alpha, \beta$.

**Lemma 7** For any $z \subseteq z'$ s.t. $z_j = z'_j$ and $x_j \cap x'_j = \phi$, $x_j \cup x'_j = z_j$

$$p(x_j|z) = p(x_j|z')$$
\[ p(x'_j|z) = p(x'_j|z') \]

**Proof.** Given the hypothesis, \( p(x_j|z) + p(x'_j|z) = p(x_j \cup x'_j|z) = 1 \) by additivity and the representation (since \( \Gamma(x_j \cup x'_j|z) = \phi \)). Similarly, \( p(x_j|z') + p(x'_j|z') = 1 \). However the Monotonicity property of structured Intuitive beliefs requires that expanding information should lead to a reduction in \( p(x_j|z) \) and \( p(x'_j|z) \). But if \( p(x_j|z) \) strictly decreases, then \( p(x'_j|z) \) must strictly increase in order to ensure that they sum to 1, which would violate the Monotonicity property. ■

This completes the argument to establish \( \tau(x_j|x_i) = 0 \) in case (c).

We have shown that for any \( x_i, x_j \) s.t. \( p(x_i, x_j|\Omega) > 0 \), it must be that \( \tau(x_j|x_i) = 0 \). Invoke Lemma 6 to establish that Positive Statistical Independence must hold.

**A.4 Proof of Proposition 2.**

**Proof.** Since there is only one relevant source of uncertainty, the likelihood of bag \( A \) relative to \( B \) is

\[
\frac{p(A|\Omega_u,n)}{p(B|\Omega_u,n)} = \exp \left[ -a(A|\Omega_u)^{-1} - a(A|n)^{-1} \right] = \exp 2a(B|n)^{-1} = \exp (2k(n-\frac{1}{2})+1).
\]

Consider the value of \( \frac{p(A,n|\Omega_u,n)}{p(B,n|\Omega_u,n)} \) at the proportion \( n = \frac{2}{3} \):

\[
\frac{p(A, \frac{2}{3}|\Omega_i, \frac{2}{3})}{p(B, \frac{2}{3}|\Omega_i, \frac{2}{3})} = \exp \left( \frac{k}{3} + 1 \right).
\]

Therefore, a higher value of \( k \) generates a higher likelihood ratio \( \frac{p(A|\Omega_i, \frac{2}{3})}{p(B|\Omega_i, \frac{2}{3})} \).

The value of \( k \) for which the agent’s posterior likelihood ratio equals the
Bayesian posterior likelihood ratio is given by
\[
\frac{p(A|\Omega_1, \frac{2}{3})}{p(B|\Omega_1, \frac{2}{3})} = \frac{0.97}{0.03} \iff k = k^* := 3 \ln \frac{0.97}{0.03} - 1 > 0,
\]
as desired.

A.5 Proof of Proposition 3

Proof. Compute that \(a(t|h) > a(h|h)\) implies
\[
\frac{p(t|hh)}{p(h|hh)} = \frac{\exp^{-1} [a(t|h)^{-1} + a(t|h)^{-1}]}{\exp^{-1} [a(h|h)^{-1} + a(h|h)^{-1}]} = \exp^{-1} 2[a(t|h)^{-1} - a(h|h)^{-1}] > 1.
\]
Moreover, \(a(t|h) = a(h|t)\) and \(a(t|t) = a(h|h)\) imply
\[
\frac{p(t|ht)}{p(h|ht)} = \frac{\exp^{-1} [a(t|h)^{-1} + a(t|t)^{-1}]}{\exp^{-1} [a(h|h)^{-1} + a(h|t)^{-1}]} = \exp^{-1} \frac{[a(h|h)^{-1} + a(h|t)^{-1}]}{[a(h|h)^{-1} + a(h|t)^{-1}]} = 1,
\]
as desired.

Finally, we show that fallacy obtains also when we allow for indirect associations:
\[
\frac{p(th|h)}{p(hh|h)} = \frac{\exp[a(t|h) + a(t|h)a(h|h)] + [a(h|h) + a(h|t)a(t|h)]}{\exp[a(h|h) + a(h|h)a(h|h)] + [a(h|h) + a(h|t)a(h|t)]}
= \exp[a(t|h) - a(h|h)][(1 + a(t|h) + a(t|h) + a(h|h)] > 1 \text{ (since } a(t|h) > a(h|h) > 0),
\]
thereby completing the proof.

References


