Impatience as Selfishness

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Introduction

- Large experimental literature on (mostly static) time preference
  - preference reversals (decreasing impatience with delay)
  - magnitude effect (decreasing impatience with magnitude)
  - non-separabilities (preference for increasing sequences, preference for spread)
  - others.


- Focus here: explanation for the magnitude effect
Introduction

• Thaler (1981): $\delta_{4000} \approx 0.75$ and $\delta_{60} \approx 0.25$

  ($3000, 0) \sim ($4000, 1 yr) and ($15, 0) \sim ($60, 1 yr)

• Loewenstein and Prelec (1992): attribute to curvature of $u$

  $(x, 0) \sim (y, t) \implies u(x) = D(t)u(y) \implies \frac{x}{y} = \frac{u^{-1}(D(t)u(y))}{y}$

• Noor (2011) calibration theorem\(^1\)

• Benhabib et al. (2010): Fixed waiting cost, $x = y - c \implies \frac{x}{y} = 1 - \frac{c}{y}$

• Ericson and Noor (2016): Magnitude effect for $(x, s) \sim (y, t)$, where $s > 0$

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\(^1\) Arbitrary discount function, concave utility, arbitrary background stream of wealth, then ($15, 0) \sim ($60, 1 yr) implies: give $60 in a year unconditionally, agent subsequently prefers $x today over $4x in a year, for any $x.
Introduction

• This paper: cognitive optimization

• Cognitive effort required for empathy
  – effortful imagination required to appreciate others’ welfare
  – also true for future (e.g. retired) self’s welfare

• Model: Current self allocates limited empathy over future selves

• Neuroscience:
  – Right temporoparietal junction linked to social cognition
  – Disable it and the agent becomes more selfish
  – Soutschek et al (2016): Also becomes more impatient

• Philosophy:
  – Hume (1739), Parfit (1971): future self is literally an other
Introduction

• Main result: Behavioral foundations = magnitude effect + regularity conditions

• Main insights:
  – Cognitive model unifies several experimental findings
  – ME as expression of self-control in multiple selves model
  – Possible link between time preference and social preference

• Other implications:
  – Justification for studying non-present biased behavior
  – Implication for extrapolation from small stakes data
  – Nonstandard behavior for large stakes due to cognitive constraints
Introduction

• Road map:

  – Introduce formal model

  – Behavioral Foundations

  – Results

  – Accommodate evidence on time preference

  – Related Literature and Concluding Comments
Model

- Consumption space $C$ – mixture space, metric topology
- Time horizon $T + 1 < \infty$
- Consumption Streams $X = C^{T+1}$, product topology, mixture space
- Take some $0 \in C$ and interpret as “base-line consumption”
- Primitive: preference $\succsim$ over $X$
Model

Definition: $\succsim$ is a Costly Empathy (CE) preference if there exists a positive regular tuple $(u, \{\varphi_t\}, K)$ such that $\succsim$ is represented by the function $U : X \to \mathbb{R}$ defined for each $x = (x_0, .., x_T) \in X$ by

$$U(x) = \sum_{t=0}^{T} D_x(t) \ u(x_t)$$

where $D_x = \text{arg} \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \ |u(x_t)| - \varphi_t(D(t)) \right\}$ s.t. $\sum_{t=0}^{T} \varphi_t(D(t)) \leq K$.

- Own utility plus utility from altruistic connections (endogenously determined)
- Cost: connection requires thought
  Benefit: utility from connection
- Increasing in $|u(x_t)|$
  - Identifying with happiest selves, concern for unhappy selves
Model

Definition: A tuple \((u, \{\varphi_t\}_{t=0}^T, K)\) is positive regular if

(i) \(u : C \to \mathbb{R}\) is continuous and mixture linear with \(u(0) = 0\) and \(u(C') = \mathbb{R}_+\)

(ii) \(\varphi_t : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}\), for each \(t\), takes the form

\[
\varphi_t(d) = \begin{cases} 
  a_t \cdot d^m & \text{if } d \leq \overline{d}_t, \\
  \infty & \text{otherwise},
\end{cases}
\]

where

\(- m > 1, a_0 = 0\) and \(\overline{d}_0 = 1\),

\(- a_t > 0\) increasing in \(t \geq 1\),

\(- 0 < \overline{d}_t \leq 1\) decreasing in \(t \geq 1\),

(iii) \(K = \infty\) or \(K = a_t \overline{d}_t^m\) for all \(t \geq 1\)

- We cannot observe \(d\) that costs more than \(K + \) can ignore boundary constraint
Model

Definition: $\succeq$ is a Costly Empathy (CE) preference if there exists a positive regular tuple $(u, \{\varphi_t\}, K)$ such that $\succeq$ is represented by the function $U : X \to \mathbb{R}$ defined for each $x = (x_0, \ldots, x_T) \in X$ by

$$U(x) = \sum_{t=0}^{T} D_x(t) \cdot u(x_t)$$

where $D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \cdot |u(x_t)| - \varphi_t(D(t)) \right\}$ s.t. $\sum_{t=0}^{T} \varphi_t(D(t)) \leq K$.

- Unconstrained model: $K = \infty$
- Constrained model: $K < \infty$
- Limit attention to positive $u$ in this talk
Model

- $U(x) = \sum_{t=0}^{T} D_x(t) \ u(x_t)$

where $D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \ |u(x_t)| - \varphi_t(D(t)) \right\}$ \quad s.t. $\sum_{t=0}^{T} \varphi_t(D(t)) \leq K.$

- $K$ is empathy cap for each individual stream in a menu

- Allocating $K$ across menu is natural, but can violate WARP
Model

- $U(x) = \sum_{t=0}^{T} D_x(t) u(x_t)$

where $D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \left| u(x_t) \right| - \varphi_t(D(t)) \right\}$ \quad s.t. \quad \sum_{t=0}^{T} \varphi_t(D(t)) \leq K.$

- Relation to hyperbolic discounting (Strotz 1955, Ainslie 1992, Laibson 1997)

- Model $D$ directly; no notion of self-control

- Here:
  - $D$ is the outcome of an explicit cognitive process
  - Temptation is $D = 0$, self-control incurs cost $\sum_{t=1}^{T} \varphi_t(D(t)) > 0$
Model

- $U(x) = \sum_{t=0}^{T} D_x(t) \ u(x_t)$

\[ \text{where } D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \ |u(x_t)| - \varphi_t(D(t)) \right\} \quad \text{s.t.} \quad \sum_{t=0}^{T} \varphi_t(D(t)) \leq K. \]

- Alternative interpretation: “[O]ur telescopic faculty is defective, and we, therefore, see future pleasures, as it were, on a diminished scale” – Pigou (1920)

- Why should lower temporal visibility reduce perceived expected pleasure?

- Interpret as the defective ability of overcoming selfishness
Model

• \( U(x) = \sum_{t=0}^{T} D_x(t) \ u(x_t) \)

\text{where } D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \ |u(x_t)| - \varphi_t(D(t)) \right\} \quad \text{s.t.} \quad \sum_{t=0}^{T} \varphi_t(D(t)) \leq K.

• Is time preference = social preference?

• Non-competitive, altruistic

• Moral imperative is weaker

• time preference \( \subset \) social preference
Model: Reduced Form

- Analyze model
- Denote a dated reward as \((c)^t\). If empathy constraint is slack, the FOC is given by:
  \[ u(c) = \phi'_t(D(t)) \]

- Discount function given by:
  \[ D_{(c)^t}(t) = \left( \frac{u(c)}{m\delta_t} \right)^{1/m-1} := \gamma(t)u(c)^{1/m-1} \]

- For large \(\lambda > 1\), empathy constraint binds for \((\lambda c)^t\), and \(D\) does not grow with \(\lambda\)
- Call \(x\) small if empathy constraint slack and large otherwise

- Intuition for behavioral identification is to look at discounting on rays of streams:
  - \(x\) is small if \(D_{\lambda x}\) is strictly increasing for all \(\lambda < 1\), otherwise large
Model: Reduced Form

• Reduced form:

**Proposition:** If \( \succcurlyeq \) admits a CE representation with \( u \geq 0 \) then \( \succcurlyeq \) is represented by

\[ U : X \rightarrow R \text{ where } \gamma(t) = (ma_t)^{-1/m} \text{ and } \]

\[ U(x) = \begin{cases} 
    u(x_0) + \sum_{t>0} \gamma(t) u(x_t)^{m} & \text{if } U(x) - u(x_0) \leq mK \\
    u(x_0) + K^m \left[ \sum_{t>0} \gamma(t) u(x_t)^{m} \right]^{m-1/m} & \text{if } U(x) - u(x_0) > mK 
\end{cases} \]

• Convenient way to determine whether a stream is small or large

• Small streams: intertemporal substitution determined by \( u \) and \( m \)

• Large streams: not additively separable, concave aggregator
Model: Foundations

- Preference $\succeq$ over $X$

**Axiom (Order).** $\succeq$ is complete and transitive.

**Axiom (Continuity).** For all $x \in X$, $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ are closed.

- Introduce $0 \in C$, and notation:
  
  $$(c)^t = (0, \ldots, 0, c_t, 0, \ldots, 0), \quad c = (c)^0$$

**Axiom (Base-line).** There exists $0 \in C$ s.t. for any $c \in C$,

$$c \sim 0 \implies (c)^t \sim 0 \text{ for all } t.$$  

- For the purpose of this talk, rule out “negative” $c$:

**Axiom (Positivity).** For all $c \in C$, $c \succeq 0$, with strict preference for some $c' \in C$. 

Model: Foundations

Axiom (Present Equivalents). For any positive $x$ there exist $c, c' \in C$ s.t.
\[ c \succeq x \succeq c'. \]

- Implies that there is no best consumption.

Axiom (Impatience). For any positive $c$ and $t < t'$, $(c)^t \succeq (c)^{t'}$.

Axiom (Risk Preference). For any $c, c', c'' \in C_0$ and $\alpha \in [0, 1]$,
\[ c \succ c' \implies \alpha c + (1 - \alpha)c'' \succ \alpha c' + (1 - \alpha)c''. \]

Axiom (Time-Invariance). For all $c, \hat{c} \in C$ and $t$,
\[ c \succeq \hat{c} \iff (c)^t \succeq (\hat{c})^t. \]

Axiom (Monotonicity). For any $x, y \in X$,
\[ (x_t, 0, \ldots, 0) \succeq (y_t, 0, \ldots, 0) \text{ for all } t \implies x \succeq y. \]

- Together these axioms are called Regularity
Model: Foundations

- Define $xy$: pays according to stream $x$ at $t$ and according to $y$ otherwise.

- For any subspace $Z \subset X$, define

**Axiom ($Z$-Separability).** For all $x \in X_s$ and all $t$,

$$\frac{1}{2}c_{xt0} + \frac{1}{2}c_{0tx} \sim \frac{1}{2}c_x + \frac{1}{2}c_0.$$

- Interpret via analogy with a hypothetical vNM pref on lotteries over streams, eg:

$$\frac{1}{2} \circ (0, c', 0) + \frac{1}{2} \circ (c, 0, c'') \sim \frac{1}{2} \circ (c, c', c'') + \frac{1}{2} \circ (0, 0, 0)$$

- Cares about distribution of consumption, not the possible correlations

- Substantive: Impatience towards a reward in one period unrelated to that in another
DU Model: Foundations

- Behaviorally define constant discounting. Write $\alpha c := \alpha c + (1 - \alpha)0$

- Constant discounting wrt scaling down by $\alpha$ if

$$ (c_x, 0, \ldots, 0) \sim (x_0, \ldots, x_T), \quad (\alpha c_x, 0, \ldots, 0) \sim (\alpha x_0, \ldots, \alpha x_T) $$

Axiom (Homotheticity).

For any $\alpha \in (0, 1]$ and $x \in X$,

$$ c_x \sim x \implies \alpha c_x \sim \alpha x. $$
DU Model: Representation

**Theorem.** A preference \(\succeq\) over \(X\) satisfies Regularity, \(X\)-Separability and Homotheticity if and only if it admits a DU Representation: there exists a regular \(u\) and weakly decreasing discount function \(D : \{0, \ldots, T\} \to [0, 1]\) with \(D(0) = 1\) such that \(\succeq\) is represented by the function \(U : X \to \mathbb{R}_+\) defined by

\[
U(x) = \sum_{t=0}^{T} D(t) u(x_t)
\]

**Theorem.** If there are two DU representations \((u^i, D^i), i = 1, 2\) of the same preference \(\succeq\), then \(D^1 = D^2\) and there exists \(\alpha > 0\) such that \(u^2 = \alpha u^1\).
Unconstrained CE Model: Foundations

- Behaviorally define magnitude effect. There is no magnitude effect if
  \[(c_x, 0, .., 0) \sim (x_0, \ldots, x_T), \quad (\alpha c_x, 0, .., 0) \sim (\alpha x_0, \ldots, \alpha x_T)\]

- There is a magnitude effect on scaling down by \(\alpha\) if
  \[(c_x, 0, .., 0) \sim (x_0, \ldots, x_T), \quad (\alpha c_x, 0, .., 0) \succ (\alpha x_0, \ldots, \alpha x_T)\]

- \(\alpha c_x \sim \alpha x\) for all \(\alpha \approx 1\) \(\implies\) no local magnitude effect at \(x\)

- \(\alpha c_x \succ \alpha x\) for all \(\alpha < 1\) \(\implies\) local magnitude effect at \(x\)

- Define the set of “small streams” \(X_s\) by
  \[X_s = \{x \in X : \alpha c_x \succ \alpha x \text{ for all } \alpha < 1\}.\]
Unconstrained CE Model: Foundations

Axiom (Weak Homotheticity).

(a) For all positive streams $x \in X$ and all $\alpha \in (0, 1)$,

$$c_x \sim x \implies \alpha c_x \gtrsim \alpha x.$$ 

(b) For any positive stream $x \in X \setminus X_s$ and any $\alpha \in (0, 1)$ s.t. $\alpha x \in X \setminus X_s$,

$$c_x \sim x \implies \alpha c_x \sim \alpha x.$$ 

(c) For any $x, y \in X_s$ s.t. $x_0 \sim y_0 \sim 0$ and any $\alpha, \beta \in (0, 1)$,

$$\beta c_x \sim \alpha x \implies \beta c_y \sim \alpha y.$$ 

- "Star-shaped" preference, $\alpha U(x) \geq U(\alpha x)$
- Homotheticity violated within $X_s$ but satisfied outside $X_s$
- Simplifying but substantive structure on $X_s$, homogeneity of magnitude effect
Unconstrained CE Model: Foundations

• Impose nonemptiness of $X_s$ and strictness of magnitude effects

**Axiom (Absorbing).** For all positive streams $x \in X \setminus C$,

(a) $\alpha c_x \succ \alpha x$ for some $\alpha \in (0, 1)$,

(b) if $\alpha c_x \succ \alpha x$ for some $\alpha \in (0, 1)$, then $\beta c_\alpha x \succ \beta \alpha x$ for all $\beta \in (0, 1)$.

• Implies: for every $x$ there is $\alpha_x$ s.t. $\alpha_x x \in X_s$ and moreover,

$$\alpha x \in X_s \implies \beta \alpha x \in X_s$$
Unconstrained CE Model: Representation

**Theorem:** A preference $\succsim$ over $X$ satisfies Regularity, X-Separability, Weak Homotheticity and Absorbing if and only if it admits an Unconstrained CE Representation: there exists a positive regular tuple $(u, (\varphi_t), K)$ with $K = \infty$ such that $\succsim$ is represented by the function $U : X \rightarrow \mathbb{R}_+$ defined by

$$U(x) = \sum_{t=0}^{T} D_x(t) \ u(x_t)$$

where $D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \ |u(x_t)| - \varphi_t(D(t)) \right\}$.

- Uniqueness presented shortly
Constrained CE Model: Foundations

- Strengthen Absorbing

**Axiom (Restricted Boundary).** *There is some \( c > 0 \) such that for all positive \( x \in X \),

\[
x \in X_s \iff (0, x_0) \preceq c.
\]

- Relax \( X \)-Separability to \( X_s \)-Separability
Constrained CE Model: Representation

**Theorem.** A preference $\succeq$ over $X$ satisfies Regularity, $X_s$-Separability, Weak Homotheticity and Restricted Boundary if and only if it admits a Constrained CE Representation: there exists a regular tuple $(u, (\varphi_t), K)$ such that $\succeq$ is represented by the function $U : X \to \mathbb{R}_+$ defined by

$$U(x) = \sum_{t=0}^{T} D_x(t) u(x_t)$$

where $D_x = \arg \max_{D \in [0,1]^{T+1}} \left\{ \sum_{t=0}^{T} D(t) \left| u(x_t) - \varphi_t(D(t)) \right| \right\}$ subject to $\sum_{t=0}^{T} \varphi_t(D(t)) \leq K$.

**Theorem.** If $(u, (\varphi_t), K)$ and $(\hat{u}, (\hat{\varphi}_t), \hat{K})$ are CMS representations for $\succeq$ then there exists $\alpha > 0$ such that $(u, (\varphi_t), K) = (\alpha \hat{u}, (\alpha \hat{\varphi}_t), \alpha \hat{K})$. 
Constrained CE Model: Proof Outline

• By $X_s$-Separability, there is an additive representation $U$ for $\succeq$ on the subdomain of “small” streams $X_s$:

$$U(x) = \sum U_t(x_t), \quad x \in X_s$$

where $U_0(\cdot)$ is linear.

• Define $u$ by $u(c) = U_0(c)$. Given Risk Preference write $U_t(x_t)$ as $U_t(u(x_t))$.

• Define $D_x$ by $D_x(t) = \frac{U_t(u(x_t))}{u(x_t)}$. Then

$$U(x) = \sum D_x(t)u(x_t), \quad x \in X_s.$$  

• By WH+Absorbing/Restricted Boundary, for each $t$, $D_x(t)$ is strictly increasing in $x$. 

Constrained CE Model: Proof Outline

- Define the additive cost function by the FOC:
  \[ u(x_t) = \varphi_t'(D_x(t)) \]
  and use first fundamental theorem of calculus to get \( \varphi_t(\cdot) \) that is strictly increasing and satisfies \( \varphi_t(0) = 0 \).
- Since \( D_x(t) \) is strictly increasing in \( x_t \) we have a strictly convex \( \varphi_t(\cdot) \).
- Maximization problem has unique solution
  \[ D_x = \arg \max_{D \in [0,1]^T} \left\{ \sum D(t)u(x_t) - \varphi_t(D(t)) \right\} \]
Constrained CE Model: Proof Outline

- Representation on $X_s$ becomes: for all $x \in X_s$

$$U(x) = \sum D_x(t)u(x_t)$$

$$s.t. \quad D_x = \arg \max_{D \in [0,1]^T} \left\{ \sum D(t)u(c_t) - \varphi_t(D(t)) \right\}$$
Constrained CE Model: Proof Outline

• Next, extend to all of $X$. For any $x \notin X_s$ define

$$\alpha_x = \max\{\alpha : \alpha x \in X_s\}.$$  

• By WH(a), for or any $x \notin X_s$,

$$c_x \sim x \implies \alpha_x c_x \sim \alpha_x x.$$  

• In any representation, $\alpha_x U(x) = U(\alpha_x x)$.

• The extended representation must satisfy

$$U(x) = \frac{1}{\alpha_x} U(\alpha_x x) = \frac{1}{\alpha_x} \sum D_{\alpha_x x}(t) u(\alpha_x x_t) = \sum D_{\alpha_x x}(t) u(x_t).$$

• That is,

$$D_x := D_{\alpha_x x}.$$
Constrained CE Model: Proof Outline

- In fact, we can always take as a representation on $X$:

$$U(x) = \sum D_x(t) u(x_t) \quad \text{s.t.}$$

$$x \in X_s \implies D_x = \arg \max_{D \in [0,1]^T} \left\{ \sum D(t) u(c_t) - \varphi_t(D(t)) \right\}$$

$$x \notin X_s \implies D_x = D_{\alpha x}.$$

- **Hurdle:** for any $x \notin X_s$, generically:

$$D_x \neq \arg \max_{D \in [0,1]^T} \left\{ \sum D(t) u(\alpha x x_t) - \varphi_t(D(t)) \right\}. $$
Constrained CE Model: Proof Outline

- The problem: Take \( x \in \text{bd}(X_s) \). Then we know
  \[
  D_x \cdot u(x) - \varphi(D_x) > D \cdot u(x) - \varphi(D) \quad \text{for all } D_x \neq D \in \Lambda
  \]

- Suppose
  \[
  D_x u(x) < D^* u(x) \text{ for some } D^* \in \Lambda.
  \]

- Then for \( \lambda > 1 \) can have
  \[
  D_x u(x) - \varphi(D_x) > D^* u(x) - \varphi(D^*)
  \]
  \[
  D_x u(\lambda x) - \varphi(D_x) < D^* u(\lambda x) - \varphi(D^*).
  \]

- Then \( D_{\lambda x} := D_x \neq \arg \max_{D \in \Lambda} \{ \sum D(t)u(\lambda x_t) - \varphi_t(D(t)) \} \)
Constrained CE Model: Proof Outline

• Need to restrict domain of maximization. The following works:

\[
\text{s.t. } D_x = \arg \max_{D \in \Lambda_x} \left\{ \sum_{t>0} D(t) \ u(x_t) - \varphi_t(D(t)) \right\}
\]

\[
\Lambda_x = \{ D \in [0, 1]^T : \sum_{t>0} \varphi_t(D(t)) \leq K_x \}
\]

• \( K_x \) satisfies \( K_x = K_{\lambda x} \) for any \( x \) and \( \lambda > 0 \)

• Establishes a very general model

• Unclear how to render \( K_x \) constant in an elegant way. So we specialize:
Constrained CE Model: Proof Outline

- Weak Homotheticity implies that \( \{ \varphi_t \} \) must be homogeneous: \( \varphi_t(d) = a_t \cdot d^m \)

- Restricted Boundary then implies \( K_x \) is independent of \( x \)

- Illustration: If \( m \) is an integer then Euler’s theorem for homogenous functions implies:
  \[
  m \sum_{t>0} \varphi_t(D_x(t)) = \sum_{t>0} D_x(t) \varphi'_t(D_x(t)) \\
  = \sum_{t>0} D_x(t) u(x_t) \\
  = U(x) - u(x_0)
  \]

- Completes the proof for the main representation
Model: Limit Results

- DU is not nested in (unconstrained) model since $X_s \neq \emptyset$. Nevertheless:

**Proposition.** Suppose that $U$ is a positive unconstrained CE representation $(u, \{\varphi_t\})$ with $\text{eff}(\varphi_t) = [0, \bar{d}_t]$.

If either

(a) $a_t \to 0$ for all $t \geq 1$, or

(b) $\bar{d}_t < 1$ for all $t \geq 1$ and $m \to \infty$,

then, for all streams $x$, the optimal discount factor $D_{u(x_t)}(t)$ satisfies

$$D_{u(x_t)}(t) = \bar{d}_t$$

for all sufficiently small $a_t$ or sufficiently large $m$, respectively.

- Cost converges to zero on eff domain, maximal discount factor $\bar{d}_t$ chosen.
Model: Limit Results

- Constrained model’s joint restriction: \( K = a_t \overline{d}_t^m \)

- \(((a_t), m, (\overline{d}_t), K)\) cannot be varied independently

- Remove \(a_t\) by setting \(a_t = K/\overline{d}_t^m\) for all \(t \geq 1\) and substitute into cost function

\[
\varphi_t(d) = K \left(\frac{d}{\overline{d}_t}\right)^m.
\]

- Capacity constraint is reduced to

\[
\sum_{t \geq 1} \left(\frac{D(t)}{\overline{d}_t}\right)^m \leq 1.
\]

- Thus, capacity constraint independent of \(K\), and \(K\) interpreted as cost parameter

- Vary \(m\) independently of \(K\) and \(\overline{d}_t\)
Model: Limit Results

Proposition. Suppose that $U$ is an Unconstrained CE representation $(u, \{\varphi_t\}, K)$ with $\text{eff}(\varphi_t) = [0, \overline{d}_t]$. Then, for all streams $x$, the optimal discount factor $D_{u(x_t)}(t)$ satisfies

$$D_{u(x_t)}(t) \to \overline{d}_t$$

as $m \to \infty$ while keeping $K$ and $\overline{d}_t$ fixed.

- $\varphi_t(d) = K \left( \frac{d}{\overline{d}_t} \right)^m$ converges to zero on the effective domain except at $\overline{d}_t$

- Discount functions arbitrarily close to $(\overline{d}_t)_{t=1}^T$ become feasible
Model: Limit Results

- Next: let $K \to 0$ and $a_t \to 0$ while keeping $\bar{d}_t$ fixed

- Cost functions vanish but capacity constraint remains the same

  \[ D = \left\{ D \in [0, 1]^{T+1} : \sum_{t \geq 1} \left( \frac{D(t)}{\bar{d}_t} \right)^m \leq 1 \right\} \]

- Constrained model converges to

  \[ U(x) = D_x \cdot u(x) \quad \text{where} \quad D_x = \arg \max_{D \in D} D \cdot u(x) \]

- Equivalent to max-max representation:

  \[ U(x) = \max_{D \in D} D \cdot u(x). \]
Application: Procrastination

• Illustrate: magnitude dependence as an instrument to influence future selves

• Time: 0, 1 and 2

• Complete two tasks before period 2: two tasks can be completed in one period.

• Each task:
  – requires immediate effort
  – yields reward in period 2, regardless of completion date

• Due to magnitude effect, self 1 would complete 2 unfinished tasks, but not 1

• Self 0 will procrastinate, even if doing both now preferred to not getting them done.
Application: Magnitude Effect

- Thaler (1981):
  
  \[(3000, 0) \sim (4000, 1 \text{ yr}) \quad \text{and} \quad (15, 0) \sim (60, 1 \text{ yr})\]

- CMS as a theory of the magnitude effect

\[D_c(t) = \gamma(t) |u(c)|^{\frac{1}{m-1}}\]
Application: Preference Reversals

- Preference reversal:
  \[ c \succ (\hat{c})^1 \quad \text{and} \quad (c)^t \prec (\hat{c})^{t+1} \quad \text{for some} \quad t > 0 \]

- Not enough empathy with self 1 and empathy for self \( t \) and \( t + 1 \) is similar

- Reverse of a preference reversal:
  \[ c \prec (\hat{c})^1 \quad \text{and} \quad (c)^t \succ (\hat{c})^{t+1} \quad \text{for some} \quad t > 0 \]

- High empathy for self 1 but not enough empathy for self \( t + 1 \) relative to self \( t \)
Application: Increasing Sequences

- Loewenstein and Prelec (1993): subjects prefer increasing sequences of consumption to constant or decreasing sequences with the same present value.

- Model: agent willing to postpone rather than prepone high consumption.

**Proposition.** Suppose $C = \mathbb{R}_+$ and $u$ is linear. For any $(c_0, c_1, c_2)$ if $c_1 < c_2$ and if $a_2/a_1$ is sufficiently close to 1, then for all $\varepsilon$ in some positive interval,

$$(c_0, c_1 + \varepsilon, c_2 - \varepsilon) \prec (c_0, c_1 - \varepsilon, c_2 + \varepsilon)$$

- Not explicable by DU model with $D < 1$
- $D_c(t)u(c)$ convex in $c$: Focus shifts in favor of higher outcomes.
- **NOTE:** under the conditions, the agent wants to concentrate consumption, and this could give a preference for decreasing sequences as well.
Application: Preference for Spread

• Define present equivalent for $c_1$ conditional on $c_2$

$$(p(c_1; c_2), 0, c_2) \sim (0, c_1, c_2)$$

Proposition. If $(0, c_1, c_2)$ is small/large, then $p(c_1; c_2)$ is constant/decreasing in $u(c_2)$.

• If $(0, c_1, c_2)$ large, then higher $c_2$ transfers empathy from self 1 to self 2

• Gives rise to “preference for spread” (Loewenstein and Prelec 1993): there exist $u(c) < u(c')$ s.t. $(0, c', c')$ large and

$$(c, 0, 0) \prec (0, c', 0)$$
$$(c, 0, c') \succ (0, c', c')$$

• Note: $c = c'$ in Loewenstein and Prelec (1993), which requires anticipatory utility
Application: Other

• **Sign effect:** more patient towards losses than gains (Hardistry et al 2013)
  
  – If $|u(-c)| > u(c)$ then we obtain

  $$D_{-c}(t) = \gamma(t) |u(-c)|^{\frac{1}{m-1}} > \gamma(t) u(c)^{\frac{1}{m-1}} = D_c(t)$$

• **Cognitive capacity:** Dohmen et al (2010) show that people with lower cognitive abilities are more impatient
  
  – Model as higher cognitive costs $\varphi$ or lower capacity $K$

• **Negative discount rates:** evidence for both positive and negative outcomes
  
  – Natural explanation requires anticipatory utility, which is absent here
Summary

• Behavioral content: Magnitude Effects + ability to rationalize findings in experiments

• Relevance for (static) time preference literature
  – Provide a theory of impatience and theory of magnitude effects
  – Stationarity violations other than preference reversals are natural

• Relevance for (dynamic) time preference literature
  – enriches multiple selves model with self-control, in the form of magnitude effects

• Behavioral content of cognitive optimization (Brunnermeier-Parker 2005, Ergin-Sarver 2010, Ellis 2015, Gabaix 2015)
  – Feasible subjective actions = effective domain of cost function
  – Here feasible discount functions $\neq$ effective domain of $\varphi$
  – Non-separability

• Left for future research: empathy in social preference, a model of time and social preference
The End