Time Preference: Experiments and Foundations*

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Abstract

In order to elicit discount functions, experiments commonly analyze how subjects trade-off money and time. However, discounting reveals itself directly only in behavior obtained by fixing the money dimension and varying the time dimension. This paper presents an experimental procedure that, in a sense made precise, better identifies the discount function. The procedure is based on a general theorem that characterizes the set of discount functions and utility indices compatible with any 'regular' preference over dated rewards. The money-time separability and curvature assumptions that are common in the literature are not required.

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1. Introduction

A common method of eliciting discount functions in the literature is based on offering subjects choices over dated rewards (m, t) and eliciting indifference points of the form

$$(s, 0) \sim (l, t),$$

where, for any date t, either the future reward l is fixed and the subjects' 'present value' s is obtained, or the present reward s is fixed and the 'future value' l is obtained.[1] The discount function $D(t)$ is elicited by computing

$$D(t) = \frac{s}{l}.$$

This method presumes that a subject's preferences $\succsim$ over dated rewards follow the *Separable Discounted Utility* (SDU) model

$$U(m, t) = D(t) \cdot u(m), \tag{1.1}$$

augmented with the assumption that u is linear. However, this method typically finds implausibly high discount rates; rates in excess of 100% are common. This has motivated a literature that seeks alternative methods of eliciting discount functions.[2] This paper belongs to this literature. It provides a novel experimental design than permits a general analysis of discounting (such as neither requiring any assumption on u nor presuming the SDU model) and, in a sense to be made precise, is based on data that reveals the underlying discount function better than most other kinds of data considered in the literature.

The required data comes from observing how subjects trade off time against time. Specifically, for a fixed small and large reward, s and l resp, it elicits for each t the delay $\Phi_l(s, t)$ such that

$$(s, t) \sim (l, \Phi_l(s, t)), \tag{1.2}$$

[1] See Fredrick et al [9] for a review of the experimental literature, and later experimental work by Coller and Williams [6] and Harrison et al [10].

[2] Andersen et al [3] replace the linearity assumption with the expected utility assumption, and they use both risk preferences and time preferences to jointly estimate several specifications of u and D. Andreoni and Sprenger [4] replace the linearity assumption with the assumption that preferences over consumption streams are represented by a time-additive SDU model $\sum D(t)u(m_t)$ with CRRA u. Subjects are asked to choose their allocation of an endowment over two periods for different interest rates and endowments, and thus their intertemporal demand curves are obtained, to which u and D are jointly fit. Attema et al [5] operationalize the idea of 'standard sequences' which circumvents the need for any assumptions on u (we describe this in Section 6 in order to compare with this paper).

that is, the subject reveals that s at any time t is as good as l at any time $\Phi_l(s,t)$. The *delay function* Φ reveals that the 'loss of attractiveness' (discounting) of s due to a delay over any interval $[t,t']$ is equal to that of l over $[\Phi_l(s,t), \Phi_l(s,t')]$, and thus it is a pure embodiment of the agent's discount function.

Our design is based on the main theoretical result in this paper, which delivers an explicit formula for computing the subject's discount function on the basis of the data Φ. Taking, for instance, the SDU model as given, the formula delivers that the subject with delay function Φ can be attributed the discount function given by

$$D(t) = e^{-r \cdot g(t)}$$

where $r > 0$ is a free parameter and g is the function that uniquely solves the functional equation (with known function Φ):

$$g(\Phi_l(s,t)) = g(t) + g(\Phi_l(s,0)).$$

The elicitation of D is then complete upto the estimation of the free parameter r. The latter can be done using any of a variety of methods used in the literature, such as by using risk preferences or preferences over streams.

To summarize, the elicitation of discount functions is separated into two tasks: obtaining the functional form of the discount function, and estimating free parameters in the functional form. The first task is achieved by adopting a design that elicits the delay function and then appling the formula delivered by the main result in this paper. The second task is achieved via additional questions (such as on how subjects rank streams of money) which enable an estimation of free parameters in the functional form obtained in the first task. The novelty is in the first task. It should be noted that the design required to obtain the delay function does not require an overhaul of usual designs. For instance, the 'multiple price lists' often used to elicit present or future values can be replaced simply with 'multiple delay lists'. Thus, the main difference is in the process of drawing conclusions from data.

The method presented in this paper opens the possibility of a more general analysis of time preference than is usually done in the literature. In order to derive the functional form for discounting we require no assumptions on the curvature of the utility u for money. Rather, we provide a formula for obtaining it from the same data. We can also take up more general representations than the SDU representation where discounting depends on the size of rewards, $D(m,t)$. This makes it possible to rigorously test the SDU model. Moreover, it allows for a rigorous examination of various hypotheses such as the *magnitude effect*, the intuitive idea that people exhibit greater patience toward larger rewards (see Fredrick et al [9] and Noor [16]). The method can also be readily adapted to analyze decision making under risk, that is, instead of eliciting information about an underlying model for preferences over money-time pairs, it can be used to elicit information about an underlying models for preferences over money-probability pairs.

The experimental design presented in this paper comes from asking the basic question: what is the *behavioral meaning* of discount functions? The data used in the common method comes from observing how subjects trade off money against time – for instance, how the present value of a reward (the money dimension) changes with t (the time dimension). But such data encodes not the discount function but rather the joint affect of discounting and the curvature of utility. We build on the intuitive idea that the data that purely expresses discounting is the behavior obtained by fixing the money dimension and varying *only* the time dimension, which is captured precisely by the delay function Φ. That is, the delay function is the behavioral meaning of discount functions. The main theoretical result is a formal validation of this assertion and a characterization of the set of discount functions consistent with a given delay function.[3]

In theoretical settings the presumption that we have all possible data makes it entirely irrelevant whether we consider present/future values or delay functions. The second main theoretical result in this paper demonstrates that limited finite data on delay functions reveals more than any arbitrary amount of finite data on present/future values. Indeed, the value of the first main result reveals itself clearly in experimental settings, where data is finite.

The remainder of the paper proceeds as follows. Section 2 presents the main theoretical result. Section 3 presents some specializations of our result obtained by imposing familiar assumptions like SDU. Section 4 presents some simulations to demonstrate how our results can be applied to uncover subjects' discount functions despite noisy data. Section 5 formally establishes our claim that limited Φ data paints a finer picture of the subject's time preference than limited money-time trade-off data, which is demonstrated in the simulations . Section 6 discusses related literature and Section 7 concludes. All proofs are contained in appendices.

[3] The basic idea of varying only one dimension while fixing others has its roots in the classic decision theory literature on multi-attribute utility and conjoint measurement theory (Fishburn [7], Krantz et al [12]). The idea arose for the purpose of axiomatizing particular utility functions, and was not operationalized in a manner that is ideal for experimental settings; see Section 6.

2. Theoretical Foundations

We present here the main theoretical result that provides the foundations for our experimental procedure.

The primitive of our analysis is a revealed preference relation $\succsim$ over the set of dated rewards $X = \mathcal{M} \times \mathcal{T}$, where time is continuous and given by $\mathcal{T} = \mathbb{R}_+$, with generic elements t, t', and the set of monetary rewards is a bounded interval $\mathcal{M} = [0, \overline{m}]$ with generic elements m, m', s, l. Data on such a preference is the minimal and simplest data that may be used to study the basic structure of time preference in a general way. In particular, it allows us to avoid confounds with orthogonal psychological considerations. For instance, if one were to consider preferences over consumption streams, then a taste for increasing consumption patterns (for which experimental evidence exists) may counteract discounting.

The choice data needed for our analysis is the *delay function* $\Phi : \mathcal{M} \times \mathcal{T} \to \mathcal{T}$, which is obtained via the indifference:

$$(m, t) \sim (\overline{m}, \Phi(m, t)),$$

for all $0 < m \leq \overline{m}$ and each t. That is, $\Phi(m, t)$ is defined as the date such that m at t is just as good as $\overline{m}$ at $\Phi(m, t)$.[4] Varying t leads to a variation in the desirability of (m, t), and this is measured by variation in the delay $\Phi(m, \cdot)$. The simplest example is a linear delay function, $\Phi(m, t) = a(m)t + b(m)$. The delay function can be obtained in practice by using the Becker-DeGroot-Marschak mechanism or by adapting the Multiple Price List (MPL) popularized by Coller and Williams [6] and Harrison et al [10].[5] Note in particular that an upheavel in existing experimental designs is not required.

Our general result presumes only that the preference $\succsim$ admits a *General Discounted Utility* (GDU) representation:

$$U(m, t) = D(m, t) \cdot u(m),$$

[4] Compared to the notation $\Phi_l(s, \cdot)$ in the Introduction for any pair of rewards $s \leq l$, here we fix the largest reward l at $\overline{m}$, and suppress it in the notation.

[5] An MPL asks questions of the form "Do you prefer \$100 now or \$$x$ in 6 months?" where x varies over a grid $x_1, ..., x_{N+1}$ of dollar amounts. The implied interest rate associated with x increases monotonically moving down the list, and the point at which the subject switches from preferring the earlier reward to the later reward determines an interval $[x_i, x_{i+1}]$ within which an indifference point '(\$100,now) $\sim$ (\$$z$,6 mth)' lies. Adapting to the current context, a 'Multiple Delay List' would ask a sequence of questions of the form "Do you prefer \$50 in 1 month or \$100 in t months?" where t varies over a range of time periods $t_1, .., t_{N+1}$ in a way that the implied interest rate decreases monotonically moving down the list.

where $D : \mathcal{M} \times \mathcal{T} \to (0, 1)$ is a *discount function* (a continuous, strictly decreasing function satisfying $D(m, 0) = 1$ and $\lim_{t\to\infty} D(m, t) = 0$ for all $m > 0$) and $u : \mathcal{M} \to \mathbb{R}_+$ is a *utility index* (a strictly increasing, continuous function satisfying $u(0) = 0$). We will often refer to the tuple (D, u) as the GDU representation.[6] If a preference admits a GDU representation we say it is *regular*. Lemma A.1 confirms that regularity corresponds only to very basic properties such completeness, transitivity, monotonicity and impatience.[7]

2.1. General Theorem

The main result in this paper identifies the *set* of discount functions attributable to the preference $\succsim$, and the utility index u that corresponds to each discount function. Say that a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a *restricted transformation* if it is continuous, strictly increasing, unbounded and satisfies $g(0) = 0$.

Theorem 2.1. *Consider a regular preference $\succsim$ and its delay function Φ. Then $\succsim$ admits the GDU representation (D, u) if and only if there is some restricted transformation g and some scalar $u(\overline{m}) > 0$ such that for all $m > 0$ and t,*

$$D(m, t) = e^{-[g(\Phi(m,t)) - g(\Phi(m,0))]},$$

and for all $m \geq 0$,

$$u(m) = e^{-g(\Phi(m,0))} \cdot u(\overline{m}).$$

The result characterizes all the discount functions and corresponding utility indices that can be attributed to the preference $\succsim$. The functional forms involve an increasing transformation g of Φ. Discount functions are defined in terms of the difference $g(\Phi(m, t)) - g(\Phi(m, 0))$, whereas utility indices are defined in terms of $g(\Phi(m, 0))$. Observe that the discount function is completely characterized in

[6] Some elementary facts about GDU representations are as follows: For any utility index u there exists a unique representation U for a regular preference $\succsim$ [8]. Each representation U can be uniquely written in the form of a GDU representation (D, u): for any representation U, the utility index u in any GDU functional form is uniquely defined by $u(m) = U(m, 0)$, and D is uniquely defined by $D(m, t) = \frac{U(m,t)}{u(m)}$ for all $m > 0$.

[7] The intuitive appeal underlying the connection between delay functions and discount functions does not rely on transitivity, so we expect that our results can be extended to models of intransitive preference, such as Ok and Masatlioglu [17]. We maintain regularity since it enables a clear exposition.

terms of the delay function Φ. While u is also characterized in terms of Φ, it essentially only reflects the information contained in present values: by definition, $(m, 0) \sim (\overline{m}, \Phi(m, 0))$. In contrast D requires information on how Φ changes as a function of t. The result reveals that obtaining a functional form for Φ is all that is necessary to obtain all the discount functions attributable to the subject. This also validates our claim that the behavioral meaning of discount functions is the delay function.

We proved the result for a class as general as the GDU class so as to enable a wide variety of applications, as we demonstrate in the sequel. Practical interest will typically be in substantially smaller subclasses with much more structure, such as the SDU class or its generalizations. The resolution of any non-uniqueness is done at the estimation stage (see below) where a specific function is finally selected.

The intuition behind the proof is as follows. Note that the two indifference points

$$(s, 0) \sim (\overline{m}, \Phi(s, 0)) \text{ and } (s, t) \sim (\overline{m}, \Phi(s, t))$$

reveal that the loss of attractiveness (due to discounting) in (s, t) relative to $(s, 0)$ must equal the loss in $(\overline{m}, \Phi(s, t))$ relative to $(\overline{m}, \Phi(s, 0))$. This translates into the statement that any discount function D attributable to the preference $\succsim$ must satisfy the equality $\frac{D(s,t)}{D(s,0)} = \frac{D(\overline{m},\Phi(s,t))}{D(\overline{m},\Phi(s,0))}$. By definition, $D(s, 0) = 1$, and so this inequality can be rewritten as:

$$D(s, t) \cdot D(\overline{m}, \Phi(s, 0)) = D(\overline{m}, \Phi(s, t)).$$

But this is a functional equation where D is the unknown function and Φ is the known function. The proof verifies that a discount function D is a solution to this functional equation if and only if there exists a utility index u for which (D, u) is a GDU representation for the preference $\succsim$. The general solution of the functional equation is leads to the statement of the theorem. (In the appendix we prove a more general result by allowing $\mathcal{M}$ to be unbounded).

2.2. Estimation

The theorem reveals how functional forms for D and u attributable to the subject can be obtained under regularity assumptions (and thus also under stronger assumptions). These functional forms will generically contain free parameters, coming from the function g which is not fully pinned down by the Φ data. For instance, in the case of the SDU model, g will only be pinned down up to a power transformation $w(g(x)) = g(x)^{\alpha}$, $\alpha > 0$, and therefore will contain exactly one free parameter.[8] The free parameters in g can be determined by exploiting any of the identifying assumptions used in the literature. Note that these require more data than preferences over dated rewards.

1. Similar to Andersen et al [3], given the utility index $u(m) = e^{-g(\Phi(m,0))}$ (normalized so that $u(\overline{m}) = 1$) and the utility function,

$$U(m,t) = D(m,t)u(m) = e^{-g(\Phi(m,t))},$$

the analyst can use both risk preference and money-time trade-off data jointly to determine the g that fits best.

2. Preferences over streams along with the assumption of additive separability produce a system of equations. That is, if preferences over streams are assumed to admit an additively separable representation with D and u:

$$U(m_0, m_1, ..) = \sum D(m_t, t)u(m_t) = \sum e^{-g(\Phi(m_t,t))},$$

then, as exploited in Attema et al [5], indifference points such as 'm_0 today and m_1 next month is as good as m'_0 today and m'_1 next month' will produce equations such as $u(m_0) + D(m_1, 1)u(m_1) = u(m'_0) + D(m'_1, 1)u(m'_1)$, that is,

$$e^{-g(\Phi(m_0,0))} + e^{-g(\Phi(m_1,1))} = e^{-g(\Phi(m'_0,0))} + e^{-g(\Phi(m'_1,1))}.$$

The g that best fits these equations can then be determined.

3. Similar to Andreoni and Sprenger [4], the expression for intertemporal demand can be computed in terms of g, and then the best fit to experimentally derived intertemporal demand curves can be sought.

4. Each of the procedures above identify D via u – this is evident when risk preferences are used and is implied by the use of money-time trade-offs when sequences are used. A possible means of identifying D without reference to u would be to determine indifference points such as 'm today is as good as the sequence yielding m at time t_1 and again at t_2' for some fixed reward m. This produces equations of the form $D(m, t_1) + D(m, t_2) = 1$, and thus

$$e^{-g(\Phi(m,t_1))} + e^{-g(\Phi(m,t_2))} = e^{-g(\Phi(m,0))}$$

[8] If (D, u) is an SDU representation then any other SDU representation $(\widehat{D}, \widehat{u})$ will satisfy $\widehat{D} = D^{\alpha}$ and $\widehat{u} = u^{\alpha}$ for some $\alpha > 0$ (this is proved in Fishburn and Rubinstein [8], for instance) Applying our theorem yields that g is identified upto a power transformation.

This concludes our discussion of estimation. This section will be the only discussion of estimation in this paper. The main contribution lies in the first step of how a functional form for discounting is obtained from data and will be the continued focus below.

3. Specializations and Applications

Our main result is proved for very general preferences specifically so that it may allow for a range of specialized results. Accordingly, we consider several assumptions of possible interest to analysts in this section.

3.1. Specialization: SDU

Suppose the analyst is interested in preferences that admit an SDU representation. Then the following proposition – a trivial corollary of our main theorem – guides the identification of the discount function. Recall that we refer to a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ as a *restricted transformation* if it is continuous, strictly increasing, unbounded and satisfies $g(0) = 0$.

Proposition 3.1. *Consider a regular preference $\succsim$ and its delay function Φ. The preference $\succsim$ admits an SDU representation with the discount function*

$$D(t) = e^{-g(t)}$$

if and only if there exists a restricted transformation g such that for all m, t,

$$g(\Phi(m,t)) - g(\Phi(m,0)) = g(t). \tag{3.1}$$

The proposition provides a necessary and sufficient condition for the existence of an SDU representations.[9] A corollary of interest for experiments is that if an SDU representation with $D(t) = e^{-g(t)}$ is assumed to exist, then g will be the function that best solves equation (3.1) *restricted to any given $m > 0$*.

As an aside, we observe that many experiments avoid making immediate payments in order to remove any immediacy effects (Coller and Williams [6], Harrison et al [10]), and the existence of a front-end delay may lead to the question

[9] The simple proof notes that a discount function $D(m,t) = e^{-[g(\Phi(m,t))-g(\Phi(m,0))]}$ in our main theorem is independent of m if and only if the power term $g(\Phi(m,t)) - g(\Phi(m,0))$ equals $f(t)$ for some function f. Observing that for $m = \overline{m}$ the power term becomes $g(t) - 0$ it follows that $f = g$. Hence the condition (3.1).

of whether the data on $\Phi(m,0)$ can at all be obtained. However, the issue and its solution are no different in our context: time 0 should be defined as any point where the analyst is willing to assume that $D(0)=1$ (or $D(\cdot,0)=1$ more generally for non-SDU models). Alternatively, one can imagine 'projecting' $\Phi(m,0)$ on the basis of the collected data. For instance, as below, if $\Phi(m,\cdot)$ is linear for $t>0$ then $\Phi(m,0)$ may be taken as the y-intercept of the linear equation.

3.2. Specialization: Non-SDU

The following proposition permits a more flexible functional form than the previous one.

Proposition 3.2. *Consider a regular preference $\succsim$ and its delay function Φ. The preference $\succsim$ admits a GDU representation with the magnitude-dependent discount function:*

$$D(m,t)=e^{-a(m)\cdot g(t)} \tag{3.2}$$

if and only if there exists a restricted transformation g and a continuous decreasing function $a(\cdot)>0$ satisfying $a(\overline{m})=1$ that together satisfy, for all m,t,

$$g(\Phi(m,t))-g(\Phi(m,0))=\ a(m)\cdot g(t). \tag{3.3}$$

The discount function (3.2) generalizes exponential discounting to permit both a non-linear treatment of time (captured by g) and a dependence of impatience on the magnitude of rewards (captured by the dependence of the discount rate $a(m)$ on the reward). This is a wide class of functional forms for D, which includes both exponential and hyperbolic discounting and their extensions – see the table below. The special case $a=1$ gives rise to the separable class $D(t)=e^{-g(t)}$ as we saw earlier.[10]

[10] It is worth noting that the restriction that $a(\cdot)$ is decreasing is an *ordinal* one, implied by basic regularity conditions. To see this, write $\delta(m)=e^{-a(m)}$, suppose $s<l$ and observe that Monotonicity requires that $\delta(l)^{f(t)}\cdot u(l)>\delta(s)^{f(t)}\cdot u(s)$ and thus $\frac{u(l)}{u(s)}>\left(\frac{\delta(s)}{\delta(l)}\right)^{f(t)}$ for all t. If $\frac{\delta(s)}{\delta(l)}>1$, this inequality cannot hold for all t, a contradiction. Thus $\delta(s)\leq\delta(l)$, and $a(\cdot)$ must be decreasing.

Φ-Function	Discount Function D	Generated by:
$\Phi(m,t) = f^{-1}(f(t) + f(\Phi_m(0)))$	$D(t) = e^{-rf(t)}$	$a = 1$
$\Phi(m,t) = (1 + \alpha\Phi_m(0))t + \Phi_m(0)$	$D(t) = (1 + \alpha t)^{-1}$	$f(t) = \ln(1 + \alpha t)$, $a = 1$
$\Phi(m,t) = a(m)t + \Phi_m(0)$	$D(m,t) = e^{-ra(m)\cdot t}$	$f(t) = t$
$\Phi(m,t) = [a(m) \cdot t^\alpha + \Phi_m(0)^\alpha]^{\frac{1}{\alpha}}$	$D(m,t) = e^{-ra(m)\cdot t^\alpha}$	$f(t) = t^\alpha$
$\Phi(m,t) = \frac{[(1+\alpha t)^{a(s)}(1+\alpha\Phi_m(0))-1]}{\alpha}$	$D(m,t) = (1 + \alpha t)^{-\varphi(m)}$	$f(t) = \ln(1 + \alpha t)$

Table 1: Φ-functions and associated D.

Unless the analyst uses richer data in order to pin down the representation (Section 2.2), there may be more than one pair of a and f satisfying (3.3) for a given Φ, and for each such pair a different representation for the preference may be obtained. While such non-uniqueness of a representation is a nuisance of sorts in theoretical work, in the current context it is a strength: different representations may provide different intuitive explanations for a given behavior, which may be relevant for interpretation and applications. The usefulness of such nonuniqueness is demonstrated below, where we showed that behavior arising from hyperbolic discounting can be replicated by the magnitude effect.

3.3. Application: Linear Φ

Suppose that the analyst finds that the data Φ is a linear function of t:[11]

$$\Phi(m,t) = a(m)t + b(m). \tag{3.4}$$

We first identify conditions under which there exists an SDU representation. Refer to m-independent discount functions $D(t)$ as *separable* discount functions.

Proposition 3.3. *Consider a regular preference $\succsim$ and with a delay function Φ that has a linear form (3.4). A separable discount function $D(t)$ can be attributed if and only if there exists $\alpha \geq 0$ such that, for all m,*

$$a(m) = 1 + \alpha b(m).$$

[11] If the agent's preference indeed admits a GDU representation, then it must be that $a(\cdot)$ and $b(\cdot)$ are decreasing, and that $a(\overline{m}) = 1$ and $b(\overline{m}) = 0$. Thus, the lines $\Phi(m,\cdot)$ are upward sloping, non-intersecting, and the curves for lower m lie strictly above those for higher m. A notable implication of the fact that $a \geq 1$ is the existence of *preference reversals* of the kind observed in intertemporal choice experiments: if $(s,0)$ is preferred to (l,d) and $(l,d+t^*)$ is preferred to (s,t^*) for some t^*, then $(l,d+t)$ is preferred to (s,t) for all $t^* \geq t$. This is satisfied vacuously if $a = 1$.

If $\alpha = 0$, then the only attributable separable discount function is exponential discounting,

$$D(t) = e^{-rt}, \qquad r > 0.$$

If $\alpha > 0$, then the only attributable separable discount function is hyperbolic discounting,

$$D(t) = (1 + \alpha t)^{-r}, \quad r > 0.$$

Thus, the test for the existence of an SDU representation is that the slopes $a(m)$ must be a linear function of the intercepts $b(m)$ in (3.4). If this condition is not satisfied, then there does not exist any SDU representation. Nevertheless, it turns out that a non-SDU representation exists:

Proposition 3.4. *The delay function Φ has a linear form (3.4) if and only if the general exponential discount function:*

$$D(m, t) = e^{-a(m) \cdot t},$$

where a is decreasing, is attributable.

Thus, when Φ is linear, it is always possible to attribute magnitude-dependent exponential discount function, whereby the discount rate $a(\cdot)$ is decreasing in the size of the reward. The fact that the subject is more patient toward larger rewards is strongly reminiscent of the *magnitude effect* (see Fredrick et al [9] for a review).

A noteworthy observation is that hyperbolic discounting is *behaviorally a special case* of the general exponential discount function on the domain of dated rewards: all subjects with linear Φ are representable by general exponential discounting but only a subset are representable by hyperbolic discounting. Therefore the analysis reveals that the magnitude effect is an alternative (and more general) explanation of the evidence – namely preference reversals – usually attributed to hyperbolic discounting. The two forms of discounting are substantially different in spirit. Hyperbolic discounting is suggestive of a self-control problem, whereas the magnitude effect is suggestive of bounded rationality: the former suggests a passion for the present [13] whereas the latter suggests that subjects pay greater attention to larger rewards [16].

4. Simulations

The purpose of this section is to demonstrate in a concrete way how Theorem 2.1 and its specializations can be used to econometrically analyze Φ data. We begin by demonstrating that money-time trade-off data can lead to substantially incorrect conclusions, and then show that our approach can uncover the true process despite noise and incorrect assumptions about errors.

4.1. Present Value Data

Suppose the utility for money is $u(m) = \ln(m+1)$, and subjects follow an exponential model $D(t) = \delta^t$. Suppose that the subjects are heterogenous in δ. Specifically, the discount factor for each subject is drawn from a uniform distribution: $\delta \sim U[0.85, 0.95]$. We suppose the analyst has 30 subjects and conducts an experiment with a 2 year horizon.

Suppose the analyst elicits the present value $p(m,t)$ of dated rewards (m,t) for $m \in \{10, 20, 30\}$ and $t \in \{6, 12, 18, 24\}$ where m is in dollars and t is in months.[12] Under the assumption of linear utility, the discount function is estimated by $\widehat{D}(m,t) = \frac{p(m,t)}{m}$. In what follows we test whether the estimated discount function satisfies the decreasing impatience property, that is, whether $\frac{\widehat{D}(m,t+6)}{\widehat{D}(m,t)} < \frac{\widehat{D}(m,t+12)}{\widehat{D}(m,t+6)}$ for $t = 0, 6, 12$ and $m = 10, 20, 30$. Write $s = 2\ln\widehat{D}(t+6) - \ln\widehat{D}(t) - \ln\widehat{D}(t+12)$ (the dependence on t and m is suppressed in the notation) and assume that s follows a normal distribution. Then the hypothesis becomes

H0 : mean$(s) = 0$

H1 : mean$(s) < 0$

The statistic is $t^* = \sqrt{n}\frac{mean(s)}{std(s)}$, which follows a standard normal distribution. The results strongly confirm the presence of decreasing impatience, although the true model is exponential.

	$t=0$		$t=6$		$t=12$	
	t^*	p	t^*	p	t^*	p
$m=10$	-15.643	0	-28.680	0	-70.377	0
$m=20$	-15.736	0	-29.778	0	-73.470	0
$m=30$	-15.760	0	-30.324	0	-74.458	0

Table 2: Hypothesis tests for decreasing impatience.

[12] The present value follows $p(m,t) = (m+1)^{\delta^t} - 1$, which depends on the realization of the random parameter δ.

Intuitively, the misspecification of the u leads to a misspecified D.

4.2. Φ Data

Above we showed how incorrect results may arise from analyzing present value data even in simple settings. Below we show that correct results arise from analyzing Φ data even when the analyst has limited data and makes incorrect assumptions about the data generating process. Specifically, the subjects will be prone to decision errors, which the analyst will not take into account. Moreover, the analyst will make incorrect normality assumptions about some distributions.

4.2.1. Models

We consider separately the three discount functions listed below. The utility for money is $u(m) = \ln(m+1)$, as before. We introduce noise into the data by assuming that some parameter in each discount function fluctuates (draws from a uniform distribution).

Exponential model: $D(t) = \delta^t$ where $\delta \sim U[0.85, 0.95]$

Hyperbolic model: $D(t) = \frac{1}{1+\alpha t}$ where $\alpha \sim U[0.65, 0.75]$

Non-SDU model: $D(m,t) = \delta^{0.92^m \cdot t^{1.5}}$ where $\delta \sim U[0.85, 0.95]$

We suppose the analyst has 30 subjects and conducts an experiment with a 2 year horizon. The subjects are prone to decision errors. Specifically, for a given subject, there is a new draw of the random parameter *every time the subject makes a decision.*

4.2.2. Results

Suppose the analyst elicits $\Phi(m,t)$ for $m \in \{10, 20\}$ and $t \in \{3, 6, 9\}$ where m is in dollars and t is in months.[13]

We first consider the *unrestricted* linear regression

$$y = b_m + c_m t, \qquad m = 10, 20, 30.$$

[13] The respective models imply the following expressions for $\Phi(m,t)$, which depend on the realization of the random parameter δ.

Exponential model: $\Phi(m,t) = t + \frac{\ln \frac{\ln m+1}{\ln 30+1}}{\ln \delta}$

Hyperbolic model: $\Phi(m,t) = \frac{\ln 30+1}{\ln m+1} t + \frac{1}{\alpha}\left(\frac{\ln 30+1}{\ln m+1} - 1\right)$

Non-SDU model: $\Phi(m,t) = \left(0.92^m t^{1.5} + \frac{1}{0.92^{30}} \frac{\ln \frac{\ln m+1}{\ln 30+1}}{\ln \delta}\right)^{\frac{1}{1.5}}$

The following table shows that the R-squared is high for the exponential and hyperbolic models, and low for the non-SDU model (we state the R-squared for the fit for each m).

	Exponential			Hyperbolic			Non-SDU		
	b_m	c_m	R^2	b_m	c_m	R^2	b_m	c_m	R^2
$m = 10$	3.273	0.972	0.926	0.479	1.333	1.000	11.404	0.187	0.179
$m = 20$	1.095	0.988	0.989	0.146	1.102	1.000	5.463	0.119	0.086
$m = 30$	0	1.000	1.000	0	1.000	1.000	0	0.189	1.000

Table 3: Results for unrestricted linear regressions

Thus Φ data for the non-SDU model is not well approximated by a linear function, while that for the exponential and hyperbolic models is. We consider nonlinear regression for the non-SDU data shortly. But first we see whether we can infer that exponential data is indeed coming from an exponential model, and similarly for the hyperbolic case. Proposition 3.3 guides us.

Exponential and Hyperbolic. Consider two nested *restricted* linear regressions. In the first, the slopes are restricted to be linear functions of the intercepts, and in the second the slopes are further restricted to equal one:

$$y = b_m + (1 + \alpha b_m)t, \quad \text{and} \quad y = c_m + t, \qquad m = 10, 20, 30.$$

The results are as follows (the R-squared is stated for the overall fit).

	First restricted regression					Second restricted regression			
	α	b_{10}	b_{20}	b_{30}	R^2	c_{10}	c_{20}	c_{30}	R^2
Exponential	-0.009	3.276	1.0852	0	0.973	2.991	1.053	0	0.973
Hyperbolic	0.695	0.479	0.146	0	1.000	1.970	0.602	0	0.971

Table 4: Results for first and second restricted linear regressions

To test whether the unrestricted linear regression explains the data better than the restricted ones, consider a likelihood ratio test. We assume that the distribution of errors is iid normal (although it is not). The formula for the likelihood ratio test is $LR = n(\ln \widetilde{\sigma}^2 - \ln \widehat{\sigma}^2)$, where $\widetilde{\sigma}^2$ and $\widehat{\sigma}^2$ are estimates of the variance of error terms under the restricted case (the null) and unrestricted case respectively and n is the sample size (number of subjects x number of time periods x number of rewards). We first test whether the unrestricted regression performs significantly better than the first restricted one (the null here), and then whether the first restricted regression performs significantly better than the second restricted one (the null in this test). The results are

	First restricted (null) vs unrestricted		Second restricted (null) vs first restricted	
	LR	p value	LR	p value
Exponential	0.196	0.375	0.156	0.282
Hyperbolic	2.560	0.278	2839	0

Table 5: Results of likelihood ratio tests for exponential and hyperbolic

Thus, we first see that for both the exponential and hyperbolic cases we cannot reasonably reject the first restricted specification in favor of the unrestricted one. That is, we cannot reject that both sets of data come from SDU models. Next we see that in the exponential case, we cannot reasonably reject the second restricted specification in favor of the first one, while in the hyperbolic case we can confidently do so. That is, given the estimates of α in Table 4, we can conclude that $\alpha \approx 0$ for the exponential case and $\alpha > 0$ for the hyperbolic case. Thus by Proposition 3.3 we (correctly) conclude that the exponential data comes from an exponential model and the hyperbolic data comes from a hyperbolic model. Note that the estimated value of α for the hyperbolic case is 0.695, whereas in the true model the parameter is randomly drawn from the interval $[0.65, 0.75]$.

Non-SDU. Next turn to evaluating the non-SDU data. Consider the unrestricted and restricted nonlinear regressions:

$$y = (b_m^\alpha + c_m t^\alpha)^{\frac{1}{\alpha}} \quad \text{and} \quad y = (b_m^\alpha + t^\alpha)^{\frac{1}{\alpha}}, \qquad m = 10, 20, 30.$$

The results are

	α	b_{10}	b_{20}	b_{30}	c_{10}	c_{20}	c_{30}	R^2
Unrestricted	2.013	11.584	5.580	0	0.520	0.160	0.035	0.907
Restricted	1.056	8.145	1.761	-3.665	–	–	–	0.596

Table 6: Results of restricted and unrestricted nonlinear regressions

Thus, the specification fits very well in its unrestricted form. A likelihood ratio test readily favors the hypothesis that the unrestricted case fits the data significantly better than the restricted case (null).

	LR	p value
Non-SDU	530.073	0

Table 7: Results of likelihood ratio tests for non-SDU

Thus we readily reject that the data comes from an SDU model. The nonlinear specification implies the correct functional form for discounting, $D(m,t) = \delta^{a(m)\cdot t^{\gamma}}$ (see Table 1). The comparison between the parameter values for the true model vs estimated model is given below.

	True	Estimated
γ	1.5	2.013
$a(10)$	0.434	0.520
$a(20)$	0.189	0.160
$a(30)$	0.081	0.035

Table 8: Comparison of true and estimated non-SDU discount function $D(m,t)=\delta^{a(m)\cdot t^{\gamma}}$

5. The Efficiency of Φ

Theorem 2.1 not withstanding, there are other ways to compute the set of compatible discount functions. For instance, it is readily seen that (D, u) is a GDU representation for a regular preference $\succsim$ if and only if u is a utility index and the discount function satisfies

$$D(m,t) = \frac{u(p(m,t))}{u(m)},$$

where $p(m,t)$ is the present value of (m,t). The proof of this is trivial, and it provides a complete characterization of the set of discount functions and corresponding utility indices just as Theorem 2.1 does. Nevertheless, Theorem 2.1 draws value from a key claim in this paper: in practical settings, where data is necessarily limited, Φ-data better reveals the information in $\succsim$ than money-time tradeoff data. We substantiate this claim here in the context of the SDU model.

Fix the set of periods and prizes and order them so that $0 = t_1 < t_2 < ... < t_J$ and $0 < m_1 < ... < m_I$. Write the corresponding finite space of dated rewards as $X_{IJ} := \{m_1, .., m_I\} \times \{0, t_2, .., t_J\}$. Suppose that the present value data is given by p_{ij} such that

$$(p_{ij}, 0) \sim (m_i, t_j) \text{ for all } m_i \text{ and all } t_j > 0.$$

Assume that the analyst is interested in SDU representations. Say that the (magnitude-independent) discount function D is *attributable* to the present value data if there exists a utility index u such that $u(p_{ij}) = D(t_j)u(m_i)$ for all $(m_i, t_j) \in$

X_{IJ}. Let the set of attributable D be denoted by $\mathcal{D}_I^p$. This is indexed by I since we will be varying I below.

Given X_{IJ}, let $p^*(= p_{I1})$ denote the present value of (m_I, t_1), the largest reward at the earliest future period. Suppose that Φ-data is obtained by determining τ_j such that

$$(p^*, \tau_j) \sim (m_I, t_j) \text{ for all } t_j > 0.$$

That is, we determine τ_j such that $\Phi(p^*, \tau_j) = t_j$. Say that the discount function D is attributable to the Φ-data if $D(t)D(\Phi(p^*, t_1)) = D(\Phi(p^*, t))$ for all these periods $\tau_1, .., \tau_J$.[14] Denote the set of attributable D by $\mathcal{D}^\Phi$.

The present value and Φ-data are related by a common time horizon t_J and also the indifference point $(p_{I1}, 0) \sim (m_I, t_1)$ which defines both p_{I1} and $\Phi(p_{I1}, 0)$. The following theorem reveals that the $J-1$ data points for Φ are more discerning than the $I \cdot (J-1)$ data points for present values, *regardless of the number I of rewards.*

Theorem 5.1. *Suppose that $\succsim$ admits some SDU representation. Then for all I,*

$$\mathcal{D}^\Phi \subset \mathcal{D}_I^p.$$

The proof of the theorem is based on the following insight: limited present value data will at best put bounds on the subjects true delay function Φ and this is the *only* extent to which it restricts the range of possible D's. The remainder of the data, no matter how rich, will only help determine what u goes with any such D (observe that in Theorem 2.1 the utility index is determined by $\Phi(\cdot, 0)$, which essentially comes from money-time trade-off data). Limited direct data on Φ will speak more than data that just puts bounds on Φ.

The 'true' D is in both $\mathcal{D}^\Phi$ and $\mathcal{D}_I^p$. The theorem therefore tells us that there is greater efficiency achieved by using Φ, in that we can get closer to the true D with fewer data points. Stated differently, the theorem reveals that the degree of potential misindentification is greater with present value data than it is with Φ-data. In this sense, limited Φ data provides a better picture of the agent's entire preference than does present value, which is the claim we set out to establish. The simulations in Section 4 illustrated this message: while theorem 5.1 reveals that by analyzing Φ the analyst's conclusions are relatively closer to the 'true' picture

[14] This is equivalent to requiring that there are utilities $0 < u(p^*) < u(m_I)$ such that $D(t)u(p^*) = D(\Phi(p^*, t))u(m_I)$ for these t's. Observe that $u(p^*) = D(\Phi(p^*, t_1))u(m_I)$ must hold and so the utilities can be substituted out, yielding the original definition.

than analyzing money-time trade-off data, the simulations provide a sense of how much closer they can be.

We close this section by noting that the theorem provides further validation of sorts for our claim that Φ serves as a behavioral definition of discount functions. It is unclear how to justify this claim in the world of infinite data since, as we noted at the start of this section, discount functions can be characterized in different ways there.

6. Related Literature

The Introduction reviewed the experimental literature on time preference. Here we discuss theoretical and experimental work that is more closely related with this paper.

The idea of looking at properties of preference over multi-dimensional objects that vary only one dimension has been used in the theoretical literatures on multi-attribute utility and conjoint measurement (Fishburn [7], Krantz et al [12]). To illustrate the way in which the idea has been used consider a preference over binary attributes (x, y). Fix any y, y' and x_0 and suppose that x_1 is a quantity such that the agent exhibits $(x_1, y) \sim (x_0, y')$. Furthermore, suppose it is determined that, iteratively for $i = 2, .., n$,

$$(x_i, y) \sim (x_{i-1}, y').$$

If the preference has a multiplicative representation, $U(x, y) = v(x) \cdot u(y)$, then each indifference point satisfies $v(x_i) \cdot u(y) = v(x_{i-1}) \cdot u(y')$, that is, for all $i = 1, .., n$,

$$\frac{v(x_i)}{v(x_{i-1})} = k$$

for some constant $k := \frac{u(y')}{u(y)}$. Since v is unique upto an affine transformation, $v(x_0)$ and $v(x_1)$ can be normalized, and consequently v is completely pinned down on $\{x_0, .., x_n\}$. When the attributes are continuous variables, the grid can be made arbitrarily finer and the entire function v can be pinned down, and similarly so can u. An analogous conclusion holds for additive utility functions and indeed with this method components of representations (like discount functions, utility functions, probability weighting functions) can be connected with behavior.

In the decision theory literature this procedure is referred to as the 'saw-tooth method' (Fishburn [7]) and the noted sequence is an example of a 'standard sequence' (Krantz et al [12]). The saw-tooth method has been used to axiomatize various models (including the exponential discounting model in Fishburn and Rubinstein [8]) and has been applied in the design of experiments on risk (Wakker and Deneffe [20] and subsequent literature) to elicit utility for money and probability weighting functions and in the design of experiments on time (Attema et al [5]) to elicit discount functions.[15]

The means of elicitation suggested in this paper differs from the saw-tooth method (as applied in a discounting context) as follows.

On a theoretical level, there are several differences. First, our method is based on the observation that the desired component of the representation can be obtained as a solution to a general functional equation, quite unlike the construction used in the saw-tooth method.[16] Second, we characterize the mapping between behavior and discount functions under minimal assumptions and do not impose the strong separability assumption that underlies additive or multiplicative representations – in fact although we maintain transitivity in this paper the intuitive connection between delay functions and discount functions does not at all hinge on it. The saw-tooth method does not work well for nonseparable representations such as $U(x,y) = v(x,y) \cdot u(y)$: each standard sequence will deliver many more unknowns than equations, and one would need to consider many standard sequences as a result. Finally, there is a foundational component in this paper – to provide a general behavioral definition of discount functions and to characterize it – whereas the saw-tooth method was not designed to single out a simple object that has intutive appeal as a general behavioral definition of discounting.

On a practical level – which, given the motivation of this paper, is a key dimension – the crucial difference is that the experimental implementation of the saw-tooth method is subject to an incentive compatibility issue, which our method is immune to. Specifically, in the sequence of questions that the analyst constructs for the subject, the answer to each question determines the questions in the rest of the sequence. For instance, the answer x_1 to the first question $(?, y) \sim (x_0, y')$ determines the next question $(?, y) \sim (x_1, y')$. Thus, by misstating preferences on earlier questions a subject can possibly increase his expected outcome from the

[15] Also related, though not on the subject of elicitation of discount functions, is Rohde [18] who exploits the idea of variation in one dimension in order to define a measure of hyperbolicity, called the 'hyperbolic factor'.

[16] To mention other related literature, specific functional equations are used by Loewenstein-Prelec [14] and Ok and Masatlioglu [17] to axiomatize specific separable discounting models. In contrast, we provide a general solution to a general functional equation.

experiment; see Harrison and Ruström [11] for further discussion of the literature based on the saw-tooth method. In our elicitation method each question is independent of the others, and thus such incentive compatibility issues are avoided altogether.

7. Concluding Remarks

This paper approaches the practical question of how to design intertemporal choice experiments from the perspective of decision theory: the behavioral foundations for the object of interest (discount functions) is studied by writing a behavioral definition and characterizing the connection between the two. The general mapping is obtained by first recognizing that the behavioral meaning of discounting must lie in the delay function Φ, and second by recognizing that functional equations provide a means of characterizing the connection between Φ and the representations (D, u). This result suggests a procedure for experimentally testing theories and eliciting discount functions that has some advantages over other procedures.

While the procedure is defined for the analysis of time preference, it applies readily to other domains as well. For instance, experiments on risk often offer subjects lotteries that have one nonzero payoff. Such lotteries can be written as (m, p), where p is the probability of the nonzero outcome. By defining 'time' as $t = \frac{1}{p} - 1$ our procedure becomes immediately applicable to the study of risk preference, where the general representation takes the form $U(m, p) = f(m, p)u(m)$ and where f is the decision weight.

A. Appendix: Regularity

Say that a preference $\succsim$ over X is *regular* if it satisfies the following basic restrictions:[17]

1- **Order**: $\succsim$ is complete and transitive.

2- **Continuity**: For each (m, t), the sets $\{(m', t') : (m', t') \succsim (m, t)\}$ and $\{(m', t') : (m, t) \succsim (m', t')\}$ are closed.

3- **Impatience**:

(i) For all $m > 0$ and $t < t'$, $(0, t) \sim (0, t')$ and $(m, t) \succ (m, t')$.

[17] Continuity presumes that $\mathbb{R}_+$ has the euclidean topology, any subset of $\mathbb{R}_+$ has the subspace topology, and X has the product topology.

(ii) For each m, m' such that $m' > m > 0$, there is t such that $(m, 0) \succ (m', t)$.
4- **Monotonicity**: For all t, if $m < m'$ then $(m', t) \succ (m, t)$.

We establish some basic results on regularity which are used later, though not always explicitly.

Lemma A.1. *For any continuous increasing $u : \mathcal{M} \to \mathbb{R}$, a regular preference $\succsim$ admits a representation $U : \mathcal{M} \times \mathcal{T} \to \mathbb{R}$ such that $U(\cdot, t)$ is continuous and strictly increasing, $U(m, \cdot)$ is continuous and strictly decreasing if $m > 0$ and constant if $m = 0$, and $U(m, 0) = u(m)$. Conversely, any preference that admits such a representation is regular. By defining $D(m, t) = \frac{U(m,t)}{u(m)}$ for any $m > 0$, any such representation can be written as a GDU representation (D, u).*

Proof. The first claim is established in [8, Thm 1]. The remaining are trivial. ■

Lemma A.2. *If $\succsim$ is regular then*

(a) For every m, t and d there exists $m' \leq m$ such that $(m', t) \sim (m, t + d)$. Moreover, for every m, t and $m' \leq m$ there exists d such that $(m', t) \sim (m, t + d)$.

(b) For any $s \leq l$ and τ such that $(s, 0) \sim (l, \tau)$, and for every $t' \geq \tau$ there exists t such that $(s, t) \sim (l, t')$. Moreover, when $s > 0$ then for any $t \geq 0$ there is a unique $T \geq t$ such that $(s, t) \sim (l, T)$.

(c) For each (m, t) there exists a unique 'present value' $\psi(m, t)$ satisfying

$$(\psi(m, t), 0) \sim (m, t).$$

Moreover, $\psi(0, \cdot) = 0$, $\psi(m, \cdot)$ is strictly decreasing for any $m > 0$, $\lim_{t\to\infty} \psi(m, t) = 0$ for all m, and $\psi(m, \cdot)$ is continuous.

(d) If $(s, 0) \sim (l, \tau)$ and $(s, t) \sim (l, T + \tau)$, then $T + \tau \geq t$.

Proof. Part (a) follows from Impatience, Monotonicity and Continuity; we omit the proof. The t in part (b) exists by Impatience, Monotonicity and Continuity: By Monotonicity, $(s, t') \precsim (l, t')$. By Impatience and the fact that $(s, 0) \sim (l, \tau)$ and $t' \geq \tau$, it follows that $(s, 0) \succsim (l, t')$. Thus, by Continuity, $(s, 0) \succsim (l, t') \succsim (s, t')$ implies that there is t such that $(s, t) \sim (l, t')$, as desired. For the second claim in (b), the existence of T is established in a similar way. Impatience guarantees uniqueness when $s > 0$.

Turning to part (c): part (a) establishes the existence of present values, and Impatience implies that $\psi(m, \cdot)$ is strictly decreasing for any $m > 0$. To see that $\lim_{t\to\infty}\psi(m,t)=0$ for all m, suppose not. Then there exists m and $s>0$ such that $(s,0)\prec(\psi(m,t),0)\sim(m,t)$ for all t. But this contradicts Impatience. Finally, to see that $\psi(m,\cdot)$ must be continuous, take any strictly increasing homeomorphism and consider the representation U delivered in Lemma A.1. Since $u(\psi(m,t))=U(m,t)$ and in particular, $\psi(m,t)=u^{-1}(U(m,t))$, continuity of u^{-1} implies that of $\psi(m,\cdot)$.

For part (d), note that if $T+\tau<t$ then $(s,T+\tau)\succ(l,T+\tau)$ by Impatience, which then violates Monotonicity. ■

The next lemma characterizes regularity in terms of properties of Φ. Say that $\Phi:\mathcal{M}\times\mathcal{T}\to\mathcal{T}$ is *generated* by $\succsim$ if for any $0<m\leq\overline{m}$ and each t, $(m,t)\sim(\overline{m},\Phi(m,t))$.

Lemma A.3. *Φ is generated by a regular preference $\succsim$ if and only if:*
(i) $\Phi(m,t)$ is continuous.
(ii) For any t, $\Phi(\cdot,t)$ is strictly decreasing and $\lim_{m\to0}\Phi(m,t)=\infty$.
(iii) For $m>0$, $\Phi(m,\cdot)$ is strictly increasing and $\Phi(\overline{m},t)=t$ for all t.

Proof. Prove the 'if' part. Let $\Phi_0^{-1}(t)$ be defined by $\Phi(\Phi_0^{-1}(t),0)=t$. Define a function $U(m,t)=\Phi_0^{-1}(\Phi(m,t))$, where the inverse exists and is continuous by the monotonicity and continuity properties in (i)-(ii). Intuitively, $U(m,t)$ is the present value of (m,t), that is, if there was a regular preference generating Φ then $(x,0)\sim(m,t)\sim(\overline{m},\Phi(m,t))$ and $(x,0)\sim(\overline{m},\Phi(x,0))$ would hold. Thus $\Phi(x,0)=\Phi(m,t)$, and in turn, $U(m,t)=x=\Phi_0^{-1}(\Phi(m,t))$.

We first verify that U represents a regular preference. By (ii), for fixed t, since $\Phi(m,t)$ strictly decreases in m and Φ_0^{-1} is also strictly decreasing, it follows that $\Phi_0^{-1}(\Phi(m,t))$ is strictly increasing in m. Therefore $U(m,t)$ is strictly increasing in m. Similarly, $U(m,t)$ is strictly decreasing in t if $m>0$. By continuity of U and by (ii), $U(0,t)=\lim_{m\to0}U(m,t)=\lim_{m\to0}\Phi_0^{-1}(\Phi(m,t))=0$ for any t. The other Impatience property follows from the fact that by (ii) and (iii), $0\leq\lim_{t\to\infty}U(m,t)=\lim_{t\to\infty}\Phi_0^{-1}(\Phi(m,t))\leq\lim_{t\to\infty}\Phi_0^{-1}(\Phi(\overline{m},t))=\lim_{t\to\infty}\Phi_0^{-1}(t)=0$, that is, $\lim_{t\to\infty}U(m,t)=0$. As already noted, U is continuous.

Finally, we check that Φ is generated by the preference $\succsim$ represented by U, that is, $U(m,t)=U(\overline{m},\Phi(m,t))$. Note that by definition and by (iii), $U(\overline{m},t)=\Phi_0^{-1}(\Phi(\overline{m},t))=\Phi_0^{-1}(t)$. Thus, $U(\overline{m},\Phi(m,t))=\Phi_0^{-1}(\Phi(m,t))=U(m,t)$, as desired. ■

B. Appendix: Proof of Thm 2.1

B.1. Proof

We formally prove the result by taking $\mathcal{M} = \mathbb{R}_+$ and noting that the same argument establishes Thm 2.1 as a corollary when $\mathcal{M} = [0, \overline{m}]$.

For a given preference $\succsim$ and any rewards $0 < s \leq l$, define the function $\Phi_{s,l}(\cdot)$ by the indifference:

$$(s, t) \sim (l, \Phi_{s,l}(t)). \tag{B.1}$$

For $s = 0 < l$, let $\Phi_{s,l}(t) := \infty$.

We first clarify the exhaustive implications of regularity on Φ.

Lemma B.1. *Φ is generated by a regular $\succsim$ if and only if:*

(i) $\Phi(s, l, t)$ is continuous,

(ii) $\Phi(s, \cdot, t)$ is strictly increasing and $\Phi(\cdot, l, t)$ is strictly decreasing in s, and moreover $\lim_{s \to 0} \Phi(s, l, t) = \infty$ when $l > 0$,

(iii) $\Phi(s, l, \cdot)$ is strictly increasing if $s, l > 0$, and $\Phi(m, m, t) = t$ for all t,

(iv) $\Phi_{m_1,m_2}(\Phi_{m_0,m_1}(t)) = \Phi_{m_0,m_2}(t)$ for all t and $m_0 \leq m_1 \leq m_2$.

Proof. Prove the 'if' part. Define $\Phi^{-1}_{(l,0)}(r)$ by $\Phi(\Phi^{-1}_{(l,0)}(r), l, 0) = r$. Let $U(m, t) := \Phi^{-1}_{m,0}(t)$, where the inverse exists by the monotonicity and continuity properties in (i)-(ii). Intuitively, $U(m, t)$ is the present value of (m, t), that is, it is a small reward $s = U(m, t)$ that satisfies,

$$\Phi(s, m, 0) = t.$$

We first verify that U represents a regular preference.

To see that U is continuous, suppose $m_n \to m$, $t_n \to t$ and to ease notation write $s_n := U(m_n, t_n)$, that is, $\Phi(s_n, m_n, 0) = t_n$. We show that s_n converges. Since $m_n \to m$, there is some M and N such that $m_n \leq M$ for all $n \geq N$ (wlog let $N = 1$). Define $T_n := \Phi(m_n, M, t_n)$. By (i), T_n converges. Observe that $\Phi(s_n, M, 0) = T_n$ by (iv). Since $\Phi(\cdot, M, 0)$ is strictly monotone and continuous, it follows that $\Phi^{-1}_{M,0}(\cdot)$ is continuous. Therefore, since T_n converges to $T := \Phi(m, M, t)$, it must be that $s_n = \Phi^{-1}_{M,0}(T_n)$ converges to $s := \Phi^{-1}_{M,0}(T) = \Phi^{-1}_{M,0}(\Phi(m, M, t))$, and in particular,

$$\Phi(s, M, 0) = \Phi(m, M, t).$$

It remains to show that $\Phi(s, m, 0) = t$. By the displayed equality and (iv), $\Phi(m, M, \Phi(s, m, 0)) = \Phi(s, M, 0) = \Phi(m, M, t)$ and so by (iii), $\Phi(s, m, 0) = t$, as desired. Thus U is continuous.

Now show the remaining regularity properties. By (ii), for fixed t, the equation $\Phi(s, m, 0) = t$ implies that as m increases, s must also increase. Therefore $U(m, t)$ is strictly increasing in m. Similarly, $U(m, t)$ is strictly decreasing in t if $m > 0$. To show the second Impatience property, take any m, m' such that $m' > m > 0$. By (ii), there is a small enough $s' > 0$ s.t. $\Phi(s', m', 0) > 0 = \Phi(m, m, 0)$. Define $t := \Phi(s', m', 0)$. Then by (ii), $U(m', t) < U(m, 0)$, as desired. Finally we show the first Impatience property, that is, $U(0, t) = 0$. By (ii) and (iii), since $\Phi(m, m, t) = t$, it must be that for s that satisfies $\Phi(s, m, 0) = t$ it must be that $s \leq m$. That is, $0 \leq U(m, t) \leq m$. Then by continuity of U, $U(0, t) = \lim_{m \to 0} U(m, t) = 0$, as desired.

To conclude, we check that Φ is generated by the preference $\succsim$ represented by U. By definition, for any $s \leq l$, $s = U(l, \Phi(s, l, 0))$. Take any t and suppose $s'' = U(l, \Phi(s, l, t))$, that is,

$$\Phi(s'', l, 0) = \Phi(s, l, t).$$

Since (iii) implies $\Phi(s, l, t) \geq \Phi(s, l, 0)$, it follows that $s'' = U(l, \Phi(s, l, t)) \leq U(l, \Phi(s, l, 0)) = s$. That is, $s'' \leq s \leq l$. By (iv), $\Phi(s, l, \Phi(s'', s, 0)) = \Phi(s'', l, 0)$ and so, by the displayed equality, $\Phi(s, l, \Phi(s'', s, 0)) = \Phi(s, l, t)$. By (iii), $\Phi(s'', s, 0) = t$, and this implies that $U(s, t) = s'' = U(l, \Phi(s, l, t))$, and thus Φ is generated by U, as desired. ■

Lemma B.2. *If D solves the functional equation (FE) below, then for any $0 < m_1 \leq m_2 \leq m_3$,*

$$D(m_2, \Phi_{m_1,m_2}(0)) \cdot D(m_3, \Phi_{m_2,m_3}(0)) = D(m_3, \Phi_{m_1,m_3}(0))$$

Proof. Suppose $m_1 \leq m_2 \leq m_3$ then the functional equation implies

$$D(m_2, \Phi_{m_1,m_2}(0)) \cdot D(m_3, \Phi_{m_2,m_3}(0)) = D(m_3, \Phi_{m_2,m_3}(\Phi_{m_1,m_2}(0)).$$

But transitivity of $\succsim$ implies $\Phi_{m_2,m_3}(\Phi_{m_1,m_2}(0)) = \Phi_{m_1,m_3}(0)$. The assertion follows. ■

Lemma B.3. *The following statements hold:*

(a) Consider any regular preference $\succsim$ *and its* Φ*-function. Then* D *can be attributed to* $\succsim$ *if and only if* D *solves the functional equation:*

$$D(s,t) \cdot D(l, \Phi_{s,l}(0)) = D(l, \Phi_{s,l}(t)), \tag{FE}$$

for all $0 < s \leq l$ *and* t.

(b) Suppose $\succsim$ *is a regular preference with the function* Φ*, and that* D *is a solution to (FE). Then* (D, u) *represents* $\succsim$ *if and only if* u *is given by*

$$u(m) = \begin{cases} D(\overline{m}, \Phi_{m,\overline{m}}(0)) \cdot u(\overline{m}) & \text{if } m \leq \overline{m} \\ [D(m, \Phi_{\overline{m},m}(0))]^{-1} \cdot u(\overline{m}) & \text{otherwise} \end{cases}, \quad \text{for all } m,$$

where $\overline{m} > 0$ *and* $u(\overline{m}) > 0$ *are arbitrary.*

Proof. We prove (a), and part (b) follows as a corollary of the proof.

First show that any attributable D must satisfy the functional equation. By regularity, $\succsim$ admits a representation U. Any representation can be written as a GDU model with some D and u. By definition of the Φ-function (1.2), it must be that for all $s, l > 0$ and t, both $u(s) = D(l, \Phi_{s,l}(0))u(l)$ and $D(s,t)u(s) = D(l, \Phi_{s,l}(t))u(l)$ hold. Rearranging yields the functional equation. Observe that we have also determined that a solution must always exist if Φ comes from a regular preference $\succsim$.

For the converse, suppose D is a solution. Take any $\overline{m} > 0$ and assign it any utility $u(\overline{m}) > 0$. Define

$$u(m) = \begin{cases} D(\overline{m}, \Phi_{m,\overline{m}}(0))u(\overline{m}) & \text{if } m \leq \overline{m} \\ D(m, \Phi_{\overline{m},m}(0))^{-1}u(\overline{m}) & \text{otherwise} \end{cases}, \quad \text{for all } m.$$

By continuity of D, the utility u is continuous as well (monotonicity will be determined shortly). Next we show that, given transitivity of $\succsim$, the utility u is consistent with D in the sense that it satisfies

$$u(s) = D(l, \Phi_{s,l}(0))u(l) \tag{B.2}$$

for all s, l s.t. $s \leq l$. To see this, consider the following cases:

Case 1- $s, l \leq \overline{m}$.

Then $u(s) = D(\overline{m}, \Phi_{s,\overline{m}}(0))u(\overline{m})$ and $u(l) = D(\overline{m}, \Phi_{l,\overline{m}}(0))u(\overline{m})$, which implies

$$u(s) = \frac{D(\overline{m}, \Phi_{s,\overline{m}}(0))}{D(\overline{m}, \Phi_{l,\overline{m}}(0))} u(l).$$

By the Lemma, $D(l,\Phi_{s,l}(0))\cdot D(\overline{m},\Phi_{l,M}(0)) = D(\overline{m},\Phi_{s,M}(0))$, that is, $\frac{D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))} = D(l,\Phi_{s,l}(0))$. It follows that (B.2) holds.

Case 2- $s \leq \overline{m} \leq l$

Then $u(s) = D(\overline{m},\Phi_{s,\overline{m}}(0))u(\overline{m})$ and $u(l) = \frac{u(\overline{m})}{D(l,\Phi_{\overline{m},l}(0))}$, which implies

$$u(s) = D(\overline{m},\Phi_{s,\overline{m}}(0))D(l,\Phi_{\overline{m},1}(0))u(l).$$

By the Lemma, $D(\overline{m},\Phi_{s,\overline{m}}(0))\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{s,l}(0))$, and (B.2) follows.

Case 3- $\overline{m} \leq s \leq l$.

Then $u(s) = \frac{u(\overline{m})}{D(s,\Phi_{\overline{m},s}(0))}$ and $u(l) = \frac{u(\overline{m})}{D(l,\Phi_{\overline{m},l}(0))}$, which implies

$$u(s) = \frac{D(l,\Phi_{\overline{m},l}(0))}{D(s,\Phi_{\overline{m},s}(0))}u(l).$$

By the Lemma, $D(s,\Phi_{\overline{m},s}(0))\cdot D(l,\Phi_{s,l}(0)) = D(l,\Phi_{\overline{m},l}(0))$, and (B.2) follows.

Thus u is consistent with D in the sense of (B.2). Observe that the equality also assures us that u must be strictly increasing: D is strictly increasing in its second argument and by Monotonicity and Impatience $\Phi_{s,l}(0)$ must be strictly increasing in s. To show that there is a GDU representation with D, define $U(m,t) := u(p(m,t))$, where $p(m,t)$ is the present value of (m,t). Since $p(m,t)$ is a representation for $\succsim$ and u is strictly increasing, it follows that $U(m,t)$ represents $\succsim$. But then $U(m,t) = u(p(m,t)) = D(m,t)u(m)$, as desired. ■

The next lemma determines how to check if D solves (FE) on the basis of information on $\Phi_{m,\overline{m}}$, $\Phi_{\overline{m},m}$ and the present value of (m,t) for all $m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)$. Write p_{mt} for the present value of (m,t), that is, $(p_{mt},0) \sim (m,t)$.

Lemma B.4. *Fix any $\overline{m} > 0$. Then D solves (FE) for all $0 < s \leq l$ and t if and only if:*

i) D solves (FE) for all s,l s.t. $0 < s \leq l$ for $s = \overline{m}$ or $l = \overline{m}$, and all t; and

ii) $D(m,t) = \frac{D(m,\Phi_{\overline{m},m}(0))}{D(p_{mt},\Phi_{\overline{m},p_{mt}}(0))}$ for all $m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)$.

Proof. The 'if' part is straightforward – note that part (ii) follows from lemma B.2. Turn to the 'only if' part. Suppose the hypothesis holds. Take any $0 < s \leq l$ and t. Consider the following cases. We make frequent use of the fact that if $m_1 \leq m_2 \leq m_3$ then transitivity implies $\Phi_{m_2,m_3}(\Phi_{m_1,m_2}(t)) = \Phi_{m_1,m_3}(t)$ for any t.

Case 1- $s,l \leq \overline{m}$.

By hypothesis, $D(s,t)\cdot D(\overline{m},\Phi_{s,\overline{m}}(0)) = D(\overline{m},\Phi_{s,\overline{m}}(t))$ and $D(l,t)\cdot D(\overline{m},\Phi_{l,\overline{m}}(0)) = D(\overline{m},\Phi_{l,\overline{m}}(t))$. Moreover, by transitivity, $\Phi_{l,\overline{m}}(\Phi_{s,l}(t)) = \Phi_{s,\overline{m}}(t)$. Observe that:

$D(l,\Phi_{s,l}(t))$

$= D(l,\Phi^{-1}_{l,\overline{m}}(\Phi_{s,\overline{m}}(t)))$ by transitivity

$= \frac{D(\overline{m},\Phi_{l,\overline{m}}[\Phi^{-1}_{l,\overline{m}}(\Phi_{s,\overline{m}}(t))])}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$ by hypothesis

$= \frac{D(\overline{m},\Phi_{s,\overline{m}}(t))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$

$= \frac{D(s,t)D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}$ by hypothesis

$= D(s,t)[\frac{D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))}]$. We are done if we show that $\frac{D(\overline{m},\Phi_{s,\overline{m}}(0))}{D(\overline{m},\Phi_{l,\overline{m}}(0))} = D(l,\Phi_{s,l}(0))$.
But this follows

since

$D(l,\Phi_{s,l}(0))D(\overline{m},\Phi_{l,\overline{m}}(0))$

$= D(\overline{m},\Phi_{l,\overline{m}}(\Phi_{s,l}(0)))$ by hypothesis

$= D(\overline{m},\Phi_{s,\overline{m}}(0))$ by transitivity. This completes the argument.

Case 2- $s \leq \overline{m} \leq l$.

By hypothesis $D(s,t)\cdot D(\overline{m},\Phi_{s,\overline{m}}(0)) = D(\overline{m},\Phi_{s,\overline{m}}(t))$ and $D(\overline{m},t)\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{\overline{m},l}(t))$ and by transitivity, $\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(t)) = \Phi_{s,l}(t)$. Observe that

$D(l,\Phi_{s,l}(t))$

$= D(l,\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(t)))$ by transitivity

$= D(\overline{m},\Phi_{s,\overline{m}}(t)) \cdot D(l,\Phi_{\overline{m},l}(0))$ by hypothesis

$= [D(s,t) \cdot D(\overline{m},\Phi_{s,\overline{m}}(0))] \cdot D(l,\Phi_{\overline{m},l}(0))$ by hypothesis

$= D(s,t) \cdot [D(\overline{m},\Phi_{s,\overline{m}}(0)) \cdot D(l,\Phi_{\overline{m},l}(0))]$. We are done if $D(\overline{m},\Phi_{s,\overline{m}}(0)) \cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{s,l}(0))$. But this follows since

$D(\overline{m},\Phi_{s,\overline{m}}(0)) \cdot D(l,\Phi_{\overline{m},l}(0))$

$= D(\overline{m},\Phi_{\overline{m},l}(\Phi_{s,\overline{m}}(0)))$ by hypothesis

$= D(\overline{m},\Phi_{s,l}(0))$ by transitivity. This completes the argument.

Case 3(i)- $\overline{m} < s \leq l$ and $t \geq \Phi_{\overline{m},s}(0)$.

By hypothesis $D(\overline{m},t)\cdot D(s,\Phi_{\overline{m},s}(0)) = D(s,\Phi_{\overline{m},s}(t))$ and $D(\overline{m},t)\cdot D(l,\Phi_{\overline{m},l}(0)) = D(l,\Phi_{\overline{m},l}(t))$ and by transitivity, $\Phi_{s,l}(\Phi_{\overline{m},s}(t)) = \Phi_{\overline{m},l}(t)$. The restriction $t \geq \Phi_{\overline{m},s}(0)$ implies that $\Phi^{-1}_{\overline{m},s}(t)$ exists. Observe that:

$D(l,\Phi_{s,l}(t))$

$= D(l,\Phi_{\overline{m},l}(\Phi^{-1}_{\overline{m},s}(t)))$ by transitivity

$= D(\overline{m},\Phi^{-1}_{\overline{m},s}(t)) \cdot D(l,\Phi_{\overline{m},l}(0))$

$= \frac{D(s,\Phi_{\overline{m},s}(\Phi^{-1}_{\overline{m},s}(t)))}{D(s,\Phi_{\overline{m},s}(0))} \cdot D(l,\Phi_{\overline{m},l}(0))$

$= D(s,t) \cdot \frac{D(l,\Phi_{\overline{m},l}(0))}{D(s,\Phi_{\overline{m},s}(0))}$. We are done if we show that $\frac{D(l,\Phi_{\overline{m},l}(0))}{D(s,\Phi_{\overline{m},s}(0))} = D(l,\Phi_{s,l}(0))$:

observe that by hypothesis and transitivity, $D(s, \Phi_{\overline{m},s}(0))D(l, \Phi_{s,l}(0)) = D(l, \Phi_{s,l}(\Phi_{\overline{m},s}(0))) = D(l, \Phi_{\overline{m},l}(0))$, as desired.

Case 3(ii)- $\overline{m} < s \leq l$ and $t < \Phi_{\overline{m},s}(0)$.

By hypothesis, $D(m,t) = \frac{D(m,\Phi_{\overline{m},m}(0))}{D(p_{mt},\Phi_{\overline{m},p_{mt}}(0))}$ where p_{mt} satisfies $(p_{mt}, 0) \sim (m, t)$. Since $t < \Phi_{\overline{m},s}(0)$, it must be that $p_{mt} \geq \overline{m}$.

$D(s,t) \cdot D(l, \Phi_{s,l}(0))$

$= \frac{D(s,\Phi_{\overline{m},s}(0))}{D(p,\Phi_{\overline{m},p}(0))} \cdot \frac{D(l,\Phi_{\overline{m}l}(0))}{D(s,\Phi_{\overline{m},s}(0))}$ by hypothesis, where p is the present value of (s, t) (s is the present value of $(l, \Phi_{s,l}(0))$ by definition)

$= \frac{D(l,\Phi_{\overline{m}l}(0))}{D(p,\Phi_{\overline{m}p}(0))}$

$= D(l, \Phi_{s,l}(t))$ by hypothesis since by transitivity p must be the present value of $(l, \Phi_{s,l}(t))$ as well.

This completes the proof. ■

Fix any $\overline{m} > 0$. For any $m > 0$ and $t \geq 0$, define $\Phi(m, t)$ by:

$$\begin{array}{ll} (m,t) \sim (\overline{m}, \Phi(m,t)) & \text{if } m \leq \overline{m} \\ (\overline{m},t) \sim (m, \Phi(m,t)) & \text{otherwise.} \end{array}$$

Lemma B.5. *D is attributable iff there is a continuous, strictly increasing and unbounded function g satisfying $g(0) = 0$ such that*

$$D(m,t) = \begin{cases} e^{-[g(\Phi(m,t))-g(\Phi(m,0))]} & \text{if } m \leq \overline{m} \\ e^{-[g(\Phi_m^{-1}(t))-g(\Phi(m,0))]} & \text{if } m \geq \overline{m} \text{ and } t \geq \Phi_{\overline{m},m}(0) \\ e^{-[g(m,0)-g(\Phi(p(m,t),0)]} & \text{if } m > \overline{m} \text{ and } t < \Phi_{\overline{m},m}(0) \end{cases}.$$

Proof. By lemma B.3, D is attributable if and only if it solves (FE). Suppose D solves (FE). Then by lemma B.3, (D, u) represents $\succsim$ for some u. Wlog, let $u(\overline{m}) = 1$. By regularity, $\Phi(m, 0)$ is continuous and strictly decreasing in m for $m < \overline{m}$ and strictly increasing for $m > \overline{m}$. Since u is strictly increasing, there is a continuous strictly increasing function g satisfying $g(0) = 0$ such that[18]

$$u(m) = \begin{cases} e^{-g(\Phi(m,0))} & \text{if } m \leq \overline{m} \\ e^{g(\Phi(m,0))} & \text{if } m \geq \overline{m} \end{cases}.$$

[18] By regularity $\Phi(\overline{m}, 0) = 0$ and so $u(\overline{m}) = e^{-g(\Phi(\overline{m},0))} = e^{-g(\Phi(\overline{m},0))} = 1$, consistent with our assumption that $u(\overline{m}) = 1$.

We will see shortly that g must be unbounded.[19] Given that (D, u) represents $\succsim$ and $u(\overline{m}) = 1$, the definition of Φ implies $u(m) = D(\overline{m}, \Phi(m, 0))$ for $m \leq \overline{m}$, and $u(m) = D(m, \Phi(m, 0))^{-1}$ for $m > \overline{m}$. Therefore,

$$\begin{array}{ll} D(\overline{m}, \Phi(m, 0)) = e^{-g(\Phi(m,0))} & \text{for } m \leq \overline{m} \\ D(m, \Phi(m, 0)) = e^{-g(\Phi(m,0))} & \text{for } m \geq \overline{m} \end{array}.$$

We use this observation below. Another observation is that by regularity $\Phi(m, 0)$ ranges from 0 to ∞ as m varies over $[0, \overline{m}]$, and so we have

$$D(\overline{m}, t) = e^{-g(t)}.$$

Moreover, since D is a discount function it must satisfy the property $\lim_{t\to\infty} D(\overline{m}, t) = 0$, which implies that g must be unbounded.

To find the general solution of (FE), we first show that D has the desired form for $0 < m \leq \overline{m}$. By the previous lemma, D solves the functional equation for s, l s.t. $[0 < s \leq l = \overline{m}]$ and all t, and in particular, it solves

$$D(m, t) \cdot D(\overline{m}, \Phi(m, 0)) = D(\overline{m}, \Phi(m, t))$$

for any $0 < m \leq \overline{m}$ and all t. Since we have determined that $D(\overline{m}, t) = e^{-g(t)}$, this functional equation therefore implies

$$D(m, t) = e^{-[g(\Phi(m,t))-g(\Phi(m,0))]} \text{ for all } m \leq \overline{m} \text{ and } t,$$

as desired.

Next consider $m \geq \overline{m}$. By the previous lemma, D must satisfy

$$D(\overline{m}, t) \cdot D(m, \Phi(m, 0)) = D(m, \Phi(m, t)).$$

Then, given the earlier observations, $D(m, \Phi(m, t)) = D(\overline{m}, t) \cdot D(m, \Phi(m, 0)) = e^{-[g(t)+g(\Phi(m,0))]}$. Therefore,

$$D(m, t) = e^{-[g(\Phi_m^{-1}(t))+g(\Phi(m,0))]} \text{ for all } m > \overline{m} \text{ and } t \geq \Phi_{\overline{m},m}(0).$$

Finally, to consider the case $[m > \overline{m}$ and $t < \Phi_{\overline{m},m}(0)]$, we note that by the previous lemma $D(m, t) = \frac{D(m, \Phi_{\overline{m},m}(0))}{D(p_{mt}, \Phi_{\overline{m},p_{mt}}(0))}$ and therefore by our earlier observations,

$$D(m, t) = e^{-[g(\Phi(m,0))-g(\Phi(p(m,t),0))]} \text{ for } m > \overline{m} \text{ and } t < \Phi_{\overline{m},m}(0).$$

[19]This does not imply that u is unbounded: though $u(m) = e^{g(\Phi(m,0))}$ for $m \geq \overline{m}$ for unbounded g, regularity does not require $\Phi(\cdot, 0)$ to be unbounded.

Thus, we have shown that if D is attributable to the preference then it must have the desired form.

To complete the proof, we need to check that the discount function solves (FE). This is straightforward to establish in light of the previous lemma. For instance, for the case where $l = \overline{m}$, we see that

$D(m,t) \cdot D(\overline{m}, \Phi(m,0)) = D(\overline{m}, \Phi(m,t))$

$\iff e^{-[g(\Phi(m,t))-g(\Phi(m,0))]} \cdot e^{-[g(\Phi(\overline{m},\Phi(m,0)))-g(\Phi(\overline{m},0))]} = e^{-[g(\Phi(\overline{m},\Phi(m,t)))-g(\Phi(\overline{m},0))]}$

$\iff e^{-[g(\Phi(m,t))-g(\Phi(m,0))+g(\Phi(\overline{m},\Phi(m,0)))-g(\Phi(\overline{m},0))]} = e^{-[g(\Phi(\overline{m},\Phi(m,t)))-g(\Phi(\overline{m},0))]}$

$\iff g(\Phi(m,t)) - g(\Phi(m,0)) + g(\Phi(\overline{m}, \Phi(m,0))) = g(\Phi(\overline{m}, \Phi(m,t)))$. But $\Phi(\overline{m}, x) := \Phi_{\overline{m},\overline{m}}(x) = x$, and thus the last equation is an identity. ■

C. Appendix: Proof of Theorem 5.1

It follows from the definition of admissibility that $D(t) = e^{-g(t)}$ is admissible for the data $\{\Phi(p_{I1}, t) : t = \tau_2, .., \tau_J\}$ if and only if g solves the functional equation

$$g(\Phi(p_{I1}, t)) = g(t) + g(\Phi(p_{I1}, 0), \qquad \text{for all } t = \tau_2, .., \tau_J.$$

The set of admissible $D(t) = e^{-g(t)}$ is nonempty (the 'true' one is in the set). Take any admissible D, and corresponding g.

Below we extend the data $\{\Phi(p_{I1}, t) : t = \tau_2, .., \tau_J\}$ to some function Φ on a subset of $X = \mathbb{R}^2_+$ in a way that is consistent with the present value data, and then proceed to prove the theorem. Specifically, we inductively define Φ on $\{p_{ij}\} \times \mathbb{R}_+$. It will be convenient to define, for each $1 \leq \iota \leq I$, the set $S_\iota \subset \{p_{ij}\}$ of all observed present values of rewards $m_\iota, .., m_I$, that is, $S_\iota := \{p_{ij}\colon \iota \leq i \leq I$ and $j = 0, .., J\}$. Note that by regularity, $m_i = p_{i0}$.

First consider $\iota = I$. Define $\Phi(m_I, t) = t$ for all t. For all j, define $\Phi(p_{Ij}, 0) = t_j$ and moreover, $\Phi(p_{Ij}, t) = g^{-1}(g(t) + g(\Phi(p_{Ij}, 0))$.

Next suppose that, for $1 < \iota \leq I$,

(a) Φ is defined for $m \in S_\iota$ and all t,

(b) $\Phi(\cdot, 0)$ is strictly increasing on S_ι,

(c) for all $p_{ij} \in S_\iota$,

$$\Phi(p_{ij}, 0) = \Phi(m_i, t_j),$$

(d) for all $m_i \in S_\iota$,

$$g(t_j) = g(\Phi(m_i, t_j)) - g(\Phi(m_i, 0))$$

Observe that this is satisfied for the case $\iota = I$ that we just defined. We now extend Φ to $S_{\iota-1}$ and all t such that these conditions are satisfied.

If $m_{\iota-1}(= p_{\iota-1,0})$ equals some $m \in S_\iota$ then define $\Phi(m_{\iota-1}, t) = \Phi(m, t)$ for all t. If $m_{\iota-1} < S_\iota$ then define $\Phi(m_{\iota-1}, 0)$ by taking any arbitrary number in $(\max_{m \in S_\iota} \Phi(m, 0), \infty)$ and let $\Phi(m_{\iota-1}, t) = g^{-1}(g(t) + g(\Phi(m_{\iota-1}, 0))$ for all t. If neither of these cases hold, then there exist $m^*, m_* \in S_\iota$ such that m^* is the smallest element in S_ι that is greater than m_ι and m_* is the largest element smaller than it. Define $\Phi(m_{\iota-1}, 0)$ by taking any number in the interval $(\Phi(m^*, 0), \Phi(m_*, 0))$ (regularity and the construction ensures that the interval is nonempty) and let $\Phi(m_{\iota-1}, t) = g^{-1}(g(t) + g(\Phi(m_{\iota-1}, 0))$ for all t. Next for all j, define $\Phi(p_{\iota-1,j}, 0) = \Phi(m_{\iota-1}, t_j)$ and moreover, $\Phi(p_{\iota-1,j}, t) = g^{-1}(g(t) + g(\Phi(p_{\iota-1,j}, 0))$. Then Φ is defined on $S_{\iota-1}$ and all t, and moreover, the analogues of (a)-(d) hold by construction. Continue this construction till we obtain Φ on $\{p_{ij}\} \times \mathbb{R}_+$. This satisfies the analogues of (a)-(d) for $\iota = 1$.

To prove the theorem, take the admissible discount function $D(t) = e^{-g(t)}$. By property (b), we can extend $\Phi(\cdot, 0)$ continuous and monotonically to all of $\mathbb{R}_+$ and define the utility index,

$$u(m) = e^{-g(\Phi(m,0))}.$$

Given properties (c) and (d), determine that, for all i, j,

$$\begin{aligned} u(p_{ij}) &= e^{-g(\Phi(p_{ij},0))} \\ &= e^{-g(\Phi(m_i,t_j))} = e^{-[g(\Phi(m_i,t_j))-g(\Phi(m_i,0))]} \cdot e^{-g(\Phi(m_i,0))} \\ &= e^{-g(t_j)} u(m_i) = D(t_j) u(m_i). \end{aligned}$$

Thus, we have shown that any D that is attributable to Φ data is also attributable to the present value data.

D. Appendix: Proof of Proposition 3.3

The proposition exploits proposition 3.1 which yields that a separable discount function $D(t) = \delta^{f(t)}$ can be attributed if and only if there exists f satisfying:

$$f(a(s)t + b(s)) = f(t) + f(b(s)).$$

First suppose such an f exists. Take any s'. Letting $t = \Phi(s', 0) = b(s')$ we see that

$$f(a(s)b(s') + b(s)) = f(b(s')) + f(b(s)) = f(a(s')b(s) + b(s')),$$

and since f is strictly increasing, $a(s)b(s') + b(s) = a(s')b(s) + b(s')$, which implies

$$\frac{a(s)-1}{b(s)} = \frac{a(s')-1}{b(s')}.$$

Thus if an f exists, then the ratio $\frac{a(s)-1}{b(s)}$ must be a constant α for all s, and so the equation $a(s) = 1 + \alpha b(s)$ must hold, as desired. To see that it must be that $\alpha \geq 0$, note when $\frac{a(s)-1}{b(s)} = \alpha$ for all s, then we have a functional equation $f((1+\alpha b(s))t + b(s)) = f(t) + f(b(s))$. Denoting $x = t$ and $y = b(s)$ we can write this as:

$$f(x + y + \alpha xy) = f(x) + f(y).$$

Suppose by way of contradiction that $\alpha < 0$. Note that for any fixed $x, y > 0$ and any $\lambda > 0$, we have $f(\lambda(x+y) - \lambda^2 |\alpha| xy) = f(\lambda x) + f(\lambda y)$. As λ increases, eventually the left hand side must decrease (since f is strictly increasing) as the right hand side increases, a contradiction. Thus $\alpha \geq 0$ must hold.

Conversely, suppose $a(s) = 1 + \alpha b(s)$ holds with $\alpha \geq 0$. Consider the above displayed functional equation. If $\alpha = 0$, then this is a cauchy functional equation and the general solution is $f(x) = cx$, $c > 0$. Then $D(t) = \delta^t$ is attributable for any $\delta \in (0,1)$. If $\alpha > 0$ then it is easily verified that $f(x) = \ln(1 + \alpha x)$ is a solution, and so for any $r > 0$, an attributable discount function is $D(t) = e^{-r\ln(1+\alpha t)} = (1 + \alpha t)^{-r}$. The uniqueness properties of the SDU model (Fishburn and Rubinstein [8]) confirm that there are no other attributable discount functions within the SDU class. ■

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