Intuitive Beliefs*

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Abstract

An agent’s intuitive beliefs over a state space are modelled as the output of a network of associations between events. Intuitive beliefs are generically nonadditive and feature non-Bayesian updating. A characterization of additivity and Bayesian updating is provided. Hypothesizing that the network is shaped by past experience, it is shown that heterogeneous intuitive beliefs may be born out of a given objective probability distribution.

JEL classification: C45, D01, D90

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1 Introduction

Investors often describe gut-feelings and intuition as important factors in decision-making after having collected and analyzed their data. The importance of such factors for real world decision-making was underscored by

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Keynes’ notion of animal spirits (Keynes 1936) and has been discussed in the management literature (Huang and Pearce 2015, Huang 2018). We maintain that incomplete deliberation must be the norm rather than the exception for agents with limited cognitive resources living in a very complex world. We find it compelling that the gaps in judgement left by incomplete deliberation are generally filled by gut feelings that reflect intuitive assessments, and therefore that intuitive assessments have a possibly significant role to play in economic behavior.

This paper seeks a formal theory of intuition. We present, first, our articulation of what intuition is, and second, the mathematical structure we use to capture it:

We take inspiration from the classic notion of associations in psychology and philosophy to provide a conceptual basis for the notion of intuition. If one hears the word “red”, one might think of a traffic signal. If one learns of someone’s vocation, ethnicity, religion, etc., one may find the image of a stereotype appear in their mind. Associations are such in-built connections between mental images of events, whereby the activation of one image activates (or inhibits) the other. Philosophers and psychologists have extensively elaborated on how associative connections can be formed through frequent or salient pairing of events and through similarities, and can be strengthened (reinforcement) or weakened (counter-conditioning or decay) over time. A central observation is that the triggering of associations is an involuntary process that entails no cognitive effort. In this regard, associative thinking is fundamentally different from deliberative reasoning. We conceptualize intuition in terms of the triggering of a chain of associations.

To formulate a mathematical model of intuitive judgements, we loosely take inspiration from neural networks in cognitive science. We model the agent as possessing an underlying network of associations between events, where the network is understood to be formed by past experience. Triggering a node in the network through some observation leads it to send signals to other nodes, in turn triggering them, and so forth. In our model, the agent’s intuitive judgement about the likelihood of an event after having observed some event is a function of the signal output of an underlying network of

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1 The philosophical school of Associationism, which dates back to the writings of Locke and Hume and served as the foundation of behavioral psychology in the early 20th century, recognized associations as the most basic function of the mind. The Associationists sought to reduce all mental life to associations, an endeavor that survived up until the cognitive revolution in the mid 20th century.
associations between events.

We analyze whether intuitive beliefs can be additive, and whether they can satisfy Bayes’ rule. We find that, if there is sufficient richness in the beliefs, then additivity and Bayesian updating each \textit{require} beliefs satisfy statistical independence for events that are deemed possible, that is, if the belief about the joint realization of variables is not zero then it must equal the product of the marginals. The results suggest that standard properties of probability measures are more likely to hold in intuitive judgements when the uncertainty faced by the agent is perceived to have a relatively simple structure.

Next we consider how intuitive beliefs may be shaped by data. Taking a system of objective conditional probability distributions (that are Bayesian) as given, we hypothesize that associative networks form such that prior intuitive marginal beliefs on pairs of events match the objective probabilities. This captures the idea that relatively simple events (co-occurrence of pairs of events) make a clearer impression on the agent’s intuitive process than more complex events, and that the process does a better job imbibing their objective probabilities.

We present our model in Sections 2 and 3, describe some of its properties in Section 4, discuss the formation of intuitive beliefs in Section 5 and consider applications of the model in Section 6. We turn to a literature review in Section 7. Section 8 concludes. Proofs are contained in the appendix.

2 Primitives and Illustration

To illustrate our model we consider an individual (“Steve”) randomly selected from North America and about whom there are two sources of uncertainty: whether he is a librarian or farmer, and his degree of introversion.\textsuperscript{2}

2.1 States

The set of \textit{sources} or \textit{elements of uncertainty} is a finite abstract set $\Gamma$ with cardinality $N > 0$ and generic elements $i, j, k, \ldots$ For each source of uncertainty $i \in \Gamma$, the abstract set $\Omega_i$, with generic elements $x_i, y_i, z_i, \ldots$, consists of all possible \textit{realizations} of source $i$, and is referred to as the \textit{elementary state}

\textsuperscript{2}This is a variation on Kahneman and Tversky’s “Steve the librarian” experiment.
space for source $i$. To avoid technicalities, we assume that each $\Omega_i$ is finite. The (full) state space is the product space given by

$$\Omega := \prod_{i \in \Gamma} \Omega_i,$$

with generic element $x = (x_1, ..., x_N)$.

In the case of Steve, the sources of uncertainty are:

$$\Gamma = \{vocation, personality\},$$

and the elementary state spaces could, for instance, be:

$$\Omega_v = \{\text{librarian, farmer}\}$$
$$\Omega_p = \{\text{shy, normal, gregarious}\},$$

and the full state space is given by:

$$\Omega = \Omega_v \times \Omega_p.$$

For instance, “Steve is a shy librarian” is a state of the world.

### 2.2 Events

An elementary event in source $i$ is a subset of $\Omega_i$, generically denoted by $x_i, y_i, z_i, \ldots$. To consider an elementary event $x_i$ is to consider the scenario where the true state lies in $x_i$ and all the states outside $x_i$ are ruled out. Fix some algebra of elementary events $\Sigma_i \subset 2^{\Omega_i}$ – for simplicity let us take this to be the power set, $\Sigma_i = 2^{\Omega_i}$. An event in the full state space is a vector of elementary events:

$$\Sigma = \prod_{i \in \Gamma} \Sigma_i.$$

A generic event is given by $x = (x_j, x_j, \ldots x_k)$.\(^3\)

\(^3\)Note that an event would normally be defined as a subset of $\Omega$ whereas we define an event as a vector of elementary events, which is a particular subset of $\Omega$. This will be convenient for our purposes as our network will have a node for every elementary event. The extension to more general events and corresponding networks is left for future research.
To illustrate, consider the information that “Steve is not gregarious”. This can be viewed as the following elementary event in the “personality” source:

\[ x_p = \{\text{shy, normal} \} \in \Sigma_p. \]

The corresponding full event, where there is no information about vocation, is

\[ (\Omega_v, x_p) \in \Sigma_v \times \Sigma_p. \]

**Notation:** For any elementary event \( x_i \in \Sigma_i \), we abuse notation and denote the full event \( x_i \Omega_{-i} \in \Sigma \) by

\[ x_i := x_i \Omega_{-i}. \]

Moreover, for any pair of events \( x, z \in \Sigma \) we define set inclusion by pair-wise set inclusion:

\[ x \subset z \iff x_i \subset z_i \text{ for all } i. \]

### 2.3 Beliefs

In our setting (where the state space has a product structure), a non-additive probability \( p(\cdot) \) over \( (\Omega, \Sigma) \) can be defined as a set function that assigns \( p(x) \in [0, 1] \) for each event \( x \in \Sigma \) and satisfies:

(i) \( p(\Omega) = 1 \),

(ii) \( p(x) = 0 \) if \( x_i = \phi \) for some \( i \),

(iii) \( x \subset y \implies p(x) \geq p(y) \).

The first condition states that regardless of the information, the agent is certain that some state is true. The second condition states that any event which specifies some empty event is deemed impossible. The third condition is a monotonicity condition.

Our primitive consists of a family of non-additive probabilities conditioned on particular events. The *prior* is given by \( p(\cdot|\Omega) \). This defines a set of ex-ante possible events

\[ \Sigma^+ = \{ z \in \Sigma : p(z|\Omega) > 0 \}. \]

Define the agent’s **beliefs** by a family of non-additive probabilities

\[ p = \{ p(\cdot|z) : z \in \Sigma \text{ s.t. } p(z|\Omega) > 0 \}, \]

that are regular in that they satisfy
(iv) \( p(x \cap z|z) = p(x|z) \) for each \((x, z) \in \Sigma \times \Sigma^+\).
The last condition captures the idea that the agent understands that her information \(z\) rules out the possibility of states outside \(z\), and so when evaluating the likelihood of event \(x\) she effectively evaluates the likelihood of \(x \cap z\).

### 2.4 Illustration of Model

To preview our model, suppose an agent (“she”) is given the information that Steve was selected randomly from North America, and is asked for her believed likelihood that Steve is a shy librarian. Dropping the braces for singleton events throughout this subsection to ease exposition, and using \(l\) to denote “librarian” and \(s\) to denote “shy”, we write her belief as:

\[
p(l, s|\Omega_v, \Omega_p).
\]

We model this agent as possessing a network of associations. Each elementary event defines a node in the network. Between any two nodes \(x_i, x_j\) there are two independent directed links with weights,

\[
a(x_i|x_j) \text{ and } a(x_j|x_i),
\]

that run from \(x_j\) to \(x_i\), and from \(x_i\) to \(x_j\), respectively.

The agent is evaluating the likelihood of the event

\[
x = (l, s).
\]

When she is presented with her prior information \(\Omega = (\Omega_v, \Omega_p)\), the nodes \(\Omega_v\) and \(\Omega_p\) are triggered. We describe what happens after, say, \(\Omega_v\) is triggered. A direct signal is sent from \(\Omega_v\) to each of the librarian and shy nodes, generating signals with respective strengths

\[
a(l|\Omega_v)^{p(l)} \text{ and } a(s|\Omega_v)^{p(s)},
\]

where \(\overline{a}(l|s) = \overline{a}(l|s) = 1\). Imagine here and below that each signal is sent sequentially (and therefore there is no issue of feedback loops).

However, the shy node is also connected to the librarian node, and sequentially sends it a signal of strength \(b_{ls}(a(s|\Omega_v))\) which depends on the strength \(a(s|\Omega_v)\) of the direct signal going into the shy node. Similarly, there
is a signal of strength \( b_{s|l}(a(l|\Omega_v)) \) going from the shy node to the librarian node. In our model these function takes the following simple form

\[
a(s|\Omega_v) \pi(l|s) \text{ and } a(l|\Omega_v) \pi(s|l).
\]

Therefore, the librarian node receives signals not only directly from \( \Omega_v \) but also indirectly through the shy node \( s \), and similarly the shy node receives a direct and indirect signal. This completes the description of what happens after \( \Omega_v \) is triggered. An analogous sequence of signals leading to output at the librarian and shy nodes is generated due to \( \Omega_p \) being triggered.

The theoretical framework for our theory is given by beliefs that aggregate these signals. A tractable formulation simply takes a product of all the signals resulting from the triggering of \( \Omega_v \) and \( \Omega_p \):

\[
p(l, s|\Omega_v, \Omega_p) = \prod_{i \in \{p, v\}} a(l|\Omega_i) \pi(l|s) \times a(s|\Omega_i) \pi(s|s) \times a(l|\Omega_i) \pi(s|l).
\]

This describes the belief \( p(l, s|\Omega_v, \Omega_p) \) that “Steve is a shy librarian”, that is, \((l, s) \in \Sigma_v \times \Sigma_p\), conditional on prior information \((\Omega_v, \Omega_p) \in \Sigma_v \times \Sigma_p\).

To illustrate the model further, suppose now that the agent receives the information that “Steve is shy”, corresponding to the event \((\Omega_v, s) \in \Sigma_v \times \Sigma_p\).

She now evaluates

\[
p(l, s|\Omega_v, s).
\]

Note that in evaluating the event (librarian, shy) there is now no uncertainty about the agent’s personality. We say that personality is now an irrelevant source of uncertainty. Since there is no uncertainty about \( s \), the model requires that there is an output of (maximum) strength 1 at the shy node (irrespective and this signal remains of strength 1 even after any indirect signals coming through the librarian node).

The terms involving output at the shy node drop out of the expression, leaving just the output at the librarian node:

\[
p(l, s|\Omega_v, s) = a(l|\Omega_v) \pi(l|l) \times a(l|s) \pi(l|l).
\]

The expression contains only the direct signals sent to the librarian node from the \( \Omega_v \) and shy nodes.

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4Equivalently the beliefs are the exponential function of the sum of logs of the signals.

5Imagine that the output at, say, the shy node \( s \) is \( \min\{1, a(s|\Omega_v) \pi(s|s) \times a(l|\Omega_v) \pi(s|l) \times a(s|s) \pi(s|s) \times a(l|s) \pi(s|l)\} \), and that the information “Steve is shy” sends an infinite large signal \( a(s|s) = \infty \) to the node \( s \).
3 Intuitive Beliefs

In this section we present our formal model. A network is defined in terms of the weights between ordered pairs of nodes. The associative network we consider will have elementary events as its nodes, and the weight between each ordered pair of nodes can be different depending on whether it is triggered directly by observation or indirectly through some other node.

3.1 Associative Network

If the agent receives information $z = (z_1, ..., z_N)$, then each node $z_i$ is triggered and (potentially) sends a signal directly to all nodes in the network. We refer to these are primary signals. The total primary signal sent to node $x_i$ from the nodes in $z = (z_1, ..., z_N)$ is given by $a(x_i|z)$, where we have allowed for generality in how the signals from triggered nodes combine to produce output at node $x_i$.

**Definition 1 [Primary Signal]** A primary signal function $a$ is a function that maps each $(x, z) \in \Sigma \times \Sigma^+$ and $i \in \Gamma$ to some

$$a(x_i|z) \in [0, 1],$$

and satisfies (i) $a(\phi_i|z) = 0$ and (ii) $a(\Omega_i|z) = 1$.

A primary signal sent to any node $\phi_i$ (corresponding to the empty elementary event in source $i$) is always 0. In contrast, the largest primary signal is sent to $\Omega_i$.

When a node $x_i$ receives a primary signal $a(x_i|z)$, it can send a secondary signal to any other node $x_j$. We assume that this secondary signal is of strength

$$a(x_i|z)^{\pi(x_i|x_j)} \in [0, 1].$$

**Definition 2 [Secondary Signal]** A secondary signal adjustment function $\pi$ is a function that maps each $x \in \Sigma$ and $i, j \in \Gamma$ to some

$$\pi(x_i|x_j) \in \mathbb{R}_+ \cup \{\infty\},$$

and satisfies $\pi(x_i|x_i) = 1$. 
Given $\overline{a}(x_i|x_i) = 1$, a primary signal $a(x_i|z)$ can be written as

$$a(x_i|z)^{\pi(x_i|x_i)}$$

and so all signals - primary and secondary - conveniently take the same form.

**Definition 3** [Network] An **associative network** is a tuple $(a, \overline{a})$ consisting of a primary signal function $a$ and a secondary signal adjustment function $\overline{a}$.

Suppose the agent is considering the likelihood of event $x$ given information $z$ and suppose $x \subset z$. It may well be that the information $z$ completely resolves uncertainty about some sources of information. For instance, when we were told that Steve is moderately introverted but not given any information about his vocation, then there was no uncertainty about his personality but there remained uncertainty about his vocation. We say that $i$ is an **irrelevant** source of uncertainty if there is no uncertainty remaining. In our model, this happens if the information $z$ sends an infinitely large signal to the $i$-elementary event $x_i$, that is, $a(x_i|z) = \infty$. Similarly, we define the set of **relevant** sources by the sources for which the uncertainty is not resolved by the information:

$$\Gamma(x|z) = \{ i \in \Gamma : a(x_i|z) < \infty \}.$$

### 3.2 Model

Adopt the convention that product over an empty set is $\prod_{i \in \emptyset} = 1$. Our model is given by:

**Definition 4** [Intuitive Beliefs] An **Intuitive Belief** representation for $p$ is an associative network $(a, \overline{a})$ such that for any $x, z \in \Sigma$ s.t. $x \subset z$,

$$p(x|z) = \prod_{(i,j) \in \Gamma(x|z)^2} a(x_j|z)^{\pi(x_i|x_j)}.$$  

To interpret, consider the likelihood $p(x|z)$. As before, primary signals take the form $a(x_j|z)^{\pi(x_j|x_j)}$ and secondary signals take the form $a(x_j|z)^{\pi(x_i|x_j)}$. The likelihood $p(x|z)$ is determined by aggregating (by multiplication) all
primary and secondary signals going into the relevant nodes, that is, into $x_i$ nodes where $i \in \Gamma(x|z)$:

$$p(x|z) = \prod_{i \in \Gamma(x|z)} a(x_i|z)^{\pi(x_i|x_j)} \prod_{j \in \Gamma} a(x_j|z)$$

Note that if $j$ is an irrelevant source of uncertainty, that is $j \notin \Gamma(x|z)$, then by definition it receives the highest primary signal $a(x_j|z) = 1$. Consequently we can replace $\prod_{j \in \Gamma}$ with $\prod_{j \in \Gamma(x|z)}$, yielding the model.

The representation is stated only for nested events $x \subset z$. By definition, beliefs satisfy $p(x \cap z|z) = p(x|z)$, and so the representation extended to non-nested events simply replaces $x$ with $x \cap z$ throughout in the right-hand side expression.$^6$

Two specializations of the model that are of interest are as follows. One special case would give the following structure to primary signals:

$$a(x_i|z) = \prod_{k \in \Gamma} a(x_i|z_k),$$

that is, each piece of source-information $z_k$ is a node that is triggered and sends a signal to a given node $x_i$, with these signals being aggregated multiplicatively as in our model. A second special case would require that secondary signal adjustment depends only on pairs of sources $a(i|j)$ rather than pairs of source-event $a(x_i|x_j)$.

4 Properties of the Model

4.1 Identification

Since $\Gamma(x_i|z) = \{i\}$, that is, when considering the marginal on $i$ the only source of uncertainty is $i$, and since by definition $a(x_i|x_i) = 1$, we see that

$$p(x_i|z) = \prod_{(i, i) \in \{i\}^2} a(x_i|z)^{\pi(x_i|x_i)} = a(x_i|z).$$

$^6$That is, for any $x, z,$

$$p(x|z) = \prod_{(i, j) \in \Gamma(x|z)^2} a(x_j \cap z_j|z)^{\pi(x_j \cap z_j|x_j \cap z_j)}.$$
That is, primary signals are identified by marginal intuitive beliefs. Consequently the model can be written as

\[ p(x|z) = \prod_{(i,j) \in \Gamma(x|z)^2} p(x_j|z)^{\pi(x_i|x_j)}. \]

A notable special case is one that satisfies Statistical Independence \( p(x|z) = \prod_{j \in \Gamma} p(x_j|z) \), which is obtained by setting \( \pi(x_i|x_j) = 0 \) for all \( i \neq j \).

Secondary signals between two nodes \( x_i, x_j \) express themselves in all beliefs – prior and updated – about the pair \( x_i, x_j \):

\[
\begin{align*}
  p(x_i, x_j|z) &= a(x_j|z)^{\pi(x_i|x_j)} a(x_j|x_i) a(x_i|z)^{\pi(x_j|x_i)} a(x_i|x_j) \\
  &= a(x_j|z)^{1+\pi(x_i|x_j)} a(x_i|z)^{1+\pi(x_j|x_i)}.
\end{align*}
\]

In particular, \( \pi(x_i|x_j) \) and \( \pi(x_j|x_i) \) must solve the linear equations

\[
(1 + \pi(x_i|x_j)) \ln a(x_j|z) + (1 + \pi(x_j|x_i)) \ln a(x_i|z) = \ln p(x_i, x_j|z)
\]

obtained by varying \( z \).

### 4.2 Updating Rule

For simplicity, consider \( x, z \) such that \( \Gamma(x|z) = \Gamma(x|\Omega) \).\footnote{In general, \( \Gamma(x|z) \subset \Gamma(x|\Omega) \) since \( i \in \Gamma(x|z) \) means \( x_i \not\subseteq z \), which implies \( x_i \not\subseteq \Omega_i \) and in turn \( i \in \Gamma(x|\Omega) \).} While the Bayesian updating rule takes the form \( p(x|z) = p(x|\Omega) \frac{p(x|z)}{p(x|\Omega)} \), the expression for our associative update rule can be readily computed to take the form

\[ p(x|z) = p(x|\Omega) \prod_{(i,j) \in \Gamma(x|z)^2} \left[ \frac{p(x_j|z)}{p(x_j|\Omega)} \right]^{\pi(x_i|x_j)}. \]

Therefore, whether information \( z \) has a positive or negative impact on the belief about \( x \) depends on how the primary signals change after information is observed.
4.3 History-Independence

If an agent learns two events then she learns their intersection. Therefore any history of events is described as a nested sequence of events $z_1 \supset z_2 \supset z_3 \ldots \supset z_n$. If this sequence terminates at $z_n$ then according to the model the agent holds beliefs $p(x|z_n)$. These beliefs rely only on the associations triggered by $z_n$ and in particular are independent of how the agent arrived at $z_n$. Therefore the model exhibits history independence.

4.4 Bayesian Intuitive Beliefs

We define

Definition 5 [Bayesian Beliefs] p is (Non-Additive) Bayesian if for any $x, z \in \Sigma$,

$$p(x|z) = \frac{p(x \cap z|\Omega)}{p(z|\Omega)}.$$

The term “Bayesian” is usually used for additive probability measures, but we use the term to describe generically non-additive beliefs that satisfy the Bayesian conditioning formula.

A natural question is whether intuitive beliefs can be Bayesian, and if so, what characterizes the intersection of the two classes of models. The main result of this section characterizes the intersection of the class of Bayesian beliefs for a class of Intuitive beliefs defined by a regularity condition: see Appendix ?? for this definition.

Say that beliefs satisfy richness if (a) $\Gamma$ is not a singleton, and (b) $p(\{x_i, x'_i\}|\Omega) < 1$ for each $i \in \Gamma$ and $x_i, x'_i \in \Omega_i$. That is, there is more than one source of uncertainty, and no binary elementary event is certain.\footnote{Since $p(\Omega_i|\Omega) = 1$, richness implies that $\Omega_i$ is not binary.}

We now show that given richness, the class of Bayesian structured Intuitive beliefs are a strict subset of the class of beliefs that satisfy a restricted form of Statistical Independence.

Theorem 1 Suppose $p$ is a regular Intuitive Belief that satisfies richness. Then $p$ is Bayesian if and only if it satisfies

(i) (Positive Statistical Independence): for all $x, z \in \Sigma$ s.t. $p(x|z) > 0$,

$$p(x|z) = \prod_{i \in \Gamma} p(x_i|z).$$
(ii) (Marginal Conditioning) for all $x_i, z$ s.t. $p(x_i | z) > 0$,

$$p(x_i | z) = \frac{p(x_i \cap z | \Omega)}{p(z_i | \Omega)}.$$ 

Positive Statistical Independence requires that beliefs about possible events are a product of their marginals. Since this is only a restriction for events that are deemed possible, Positive Statistical Independence permits the possibility that all marginals are positive but the joint belief is zero. Marginal Conditioning embodies a form of Bayesian conditioning of marginal beliefs augmented with the property that the updating of marginals on source $i$ is independent of the information about other sources.\(^9\) It is noteworthy that while Bayesian updating is a relationship among all conditional beliefs, it imposes a strong restriction (namely Positive Statistical Independence) on each conditional belief. This is a reflection of the fact that in the world of Intuitive beliefs, conditional preferences are not independent of each other: conditional beliefs $p(\cdot | z)$ for each $z$ all share the same function $a$.

An upshot of this result is that it is in fact possible for Intuitive beliefs to be Bayesian, albeit only when beliefs have a particularly simple structure. Presumably if the uncertainty perceived by the agent is in some sense complex, then she is not likely to be Bayesian.

The theorem requires that there exist at least two sources of uncertainty. When there is only one source of uncertainty, the model takes the form $p(x_i | z_i) = a(x_i | z_i)$ and clearly lacks any kind of structure. Indeed, Bayesian Intuitive beliefs are simply Bayesian beliefs when there is only one source of uncertainty.

### 4.5 Additive Intuitive Beliefs

Next turn to the characterization of additivity. The usual definition of additivity does not apply in our setting because $p$ is defined on vectors of events, that is, $\Sigma = \prod_{i \in \Gamma} \Sigma_i$. Here $\Sigma_i$ is required to be an algebra, and not $\Sigma$. In \(^9\)It is straightforward to show how this condition along with Positive Statistical Independence yields Bayesian updating of possible events: $p(x | z) = \prod_{i \in \Gamma} p(x_i | z) = \prod_{i \in \Gamma} \frac{p(x_i \cap z_i | \Omega)}{p(z_i | \Omega)} = \frac{\prod_{i \in \Gamma} p(x_i \cap z_i | \Omega)}{\prod_{i \in \Gamma} p(z_i | \Omega)} = \frac{p(x \cap z | \Omega)}{p(z | \Omega)}.$
particular, the union $x \cup x'$ of two events $x$ and $x'$ in $\Sigma$ may not belong to $\Sigma$. However, if we have events of the form $x_i x_{-i}$ and $x'_i x_{-i}$, that is, events that differ only on one source, then there exists a point-wise union of these events in $\Sigma$, denoted $x_i \cup x'_i x_{-i}$. We exploit this to define:

**Definition 6** \[\text{Additivity}\] $p$ satisfies **Source Additivity** if all $x, x', z \in \Sigma$ and $i \in \Gamma$ s.t. $x_i \cap x'_i = \phi$, 

$$p(x_i \cup x'_i, x_{-i} | z) = p(x_i, x_{-i} | z) + p(x'_i, x_{-i} | z).$$

$p$ satisfies **Marginal Additivity** if all $x, x', z \in \Sigma$ and $i \in \Gamma$ s.t. $x_i \cap x'_i = \phi$, 

$$p(x_i \cup x'_i | z) = p(x_i | z) + p(x'_i | z).$$

Marginal Additivity is the generalization of Source Additivity that applies only to marginals. Our next result requires a richness condition on beliefs. Say that $p$ satisfies \textit{richness}$^*$ if there are at least 3 distinct sources, and for any $x_i$ s.t. $0 < p(x_i | \Omega_i) < 1$ and any $z, z'$ s.t. $x_i \subset z_i = z'_i$,

$$z \not\subseteq z' \implies p(x_i | z) \neq p(x_i | z').$$

That is, for any $x_i$ that is possible and uncertain and is contained by current information, decreasing the available information always changes the conditional belief about $x_i$.

Richness$^*$ is required only for the sufficiency part of the following result.

**Theorem 2** Consider a regular Intuitive Belief $p$ that satisfies \textit{richness}$^*$. Then $p$ satisfies Source Additivity iff it satisfies Marginal Additivity and Positive Statistical Independence.

The result tells us that, like Bayesian updating, Source Additivity also imposes Positive Statistical Independence on Intuitive beliefs.

Put together we see that within the context of Intuitive beliefs, the class of Bayesian beliefs and the class of Source Additive beliefs are overlapping and distinct, which is implied by the fact that the updating and additivity restrictions implied by each are not mutually exclusive. Moreover, both are (strict) subsets of the class of beliefs satisfying Positive Statistical Independence. A (potentially testable) implication is that if beliefs do not satisfy Positive Statistical Independence, then non-Bayesian updating and non-Additivity must occur simultaneously.
5 Formation of Intuitive Beliefs

Having formulated the structure of associative networks, we proceed to hypothesize how the weights between each ordered link are determined. Taking as a starting point that associations between events are built from experience, we posit that intuitive beliefs $p$ are shaped by a system of objective probability distributions:

- For any event $z \in \Sigma$, there is an (additive) probability distribution $q(\cdot|z)$ over $(\Omega, \Sigma)$
- The system of conditional distributions is Bayesian.

We hypothesize that intuitive beliefs are trained by relatively simple events, capturing the idea that simpler events are more noticeable and more readily processed than complex ones. Since marginals on pairs $x_i, x_j$ are sufficient to capture statistical dependence, we assume that the network forms in a way to try to match prior objective probabilities $q(x_i|x_j|\Omega)$:

**Definition 7 (Trained Beliefs)** An intuitive belief $p$ is trained by $q$ if the parameters $(a, \bar{a})$ of the corresponding network solve:

$$\min_{a, \bar{a}} \max_{x_i, x_j} |q(x_i|x_j|\Omega) - p(x_i|x_j|\Omega)|$$

and for each $\Omega \neq z \in \Sigma$,

$$\min_{a, \bar{a}} \max_{x_i} |q(x_i|z) - p(x_i|z)|$$

Implicit here is that the agent has limited market experience in that she is not aware of any information. A stronger hypothesis (which we do not pursue here) is that the agent observes information and has sufficient market experience that her network seeks to match conditional objective probabilities:

$$\min_{a, \bar{a}} \max_{x_i, x_j, z} |q(x_i|x_j|z) - p(x_i|x_j|z)|$$

We assume that the agent has the cognitive resources to update marginal beliefs in a Bayesian fashion.

**Assumption 1** For all $x_i, z \in \Sigma$, marginal intuitive beliefs $p(x_i|z)$ are Bayesian updates of prior intuitive beliefs $p(x_i|\Omega)$.
5.1 Results

Our first result is that intuitive beliefs are flexible enough to completely match the objective probabilities of simple events, that is, the problems in minimization problem posited above can be solved to obtain:

**Proposition 1** For each \((x_i, x_j) \in \Sigma_i \times \Sigma_j\) and \(z \in \Sigma^+\),

(i) \(q(x_i, x_j | \Omega) = p(x_i, x_j | \Omega)\), and

(ii) \(q(x_i | z) = p(x_i | z)\).

**Proof.** For any \(x_i \in \Sigma\), let \(a(x_i | \Omega) = q(x_i | \Omega)\). Since, by assumption, both intuitive and objective marginals are Bayesian updates, we have \(p(x_i | z) = a(x_i | z) = q(x_i | z)\) for all \(x_i, z \in \Sigma\), that is, the marginals match. Next, determine \(\overline{q}(x_i | x_j)\) and \(\overline{q}(x_j | x_i)\) by

\[
\min_{\overline{q}(x_i | x_j), \overline{q}(x_j | x_i)} |q(x_i, x_j | \Omega) - p(x_i, x_j | \Omega)|.
\]

Notice that

\[
\min_{\overline{q}(x_i | x_j), \overline{q}(x_j | x_i)} |q(x_i, x_j | \Omega) - p(x_i, x_j | \Omega)|
= \min_{\overline{q}(x_i | x_j), \overline{q}(x_j | x_i)} |q(x_i, x_j | \Omega) - p(x_i, x_j | \Omega)^{1+\overline{q}(x_i | x_j)} p(x_j | \Omega)^{1+\overline{q}(x_j | x_i)}| 
= \min_{\overline{q}(x_i | x_j), \overline{q}(x_j | x_i)} |q(x_i, x_j | \Omega) - q(x_i | \Omega)^{1+\overline{q}(x_i | x_j)} q(x_j | \Omega)^{1+\overline{q}(x_j | x_i)} |
\]

We see therefore that this minimum can equal 0 for any \((\overline{a}(x_i | x_j), \overline{a}(x_j | x_i)) \in \mathbb{R}_+^2\) that solves

\[
q(x_i, x_j | \Omega) = q(x_i | \Omega)^{1+\overline{q}(x_i | x_j)} q(x_j | \Omega)^{1+\overline{q}(x_j | x_i)}.
\]

This completes the proof. \(\blacksquare\)

Define the set

\[
\overline{\Gamma}(x_i, x_j) = \{ (\alpha, \beta) \in \mathbb{R}_+^2 : q(x_i, x_j | \Omega) = q(x_i | \Omega)^{1+\alpha} q(x_j | \Omega)^{1+\beta} \}
\]

As a corollary of our previous result we obtain a characterization of intuitive beliefs that are trained by \(q\):

**Proposition 2** An intuitive belief \(p\) is trained by \(q\) if and only if for each \(x, z\),

\[
p(x | z) = \prod_{j \in \Gamma} q(x_j | z)^{\sum_{i \in \Gamma(x_i | z)} \overline{q}(x_i | x_j)}
\]

s.t. \((\overline{a}(x_i | x_j), \overline{a}(x_j | x_i)) \in \overline{\Gamma}(x_i, x_j)\)

The characterization yields that two agents may arrive at different intuitive beliefs despite facing the same experiences. Therefore our model endogeneously generates heterogeneous beliefs.
5.2 Refinement

We propose two ways of selecting a solution. To obtain a one-to-one mapping between $p$ and $q$ then it would be natural to focus on the symmetric solution, where for any $x_i, x_j$, we have $\overline{a}(x_i|x_j) = \overline{a}(x_j|x_i)$. Write these as $\overline{a}(x_i,x_j)$. For any $i \neq j$, the symmetric weights are given by

$$\overline{a}(x_i,x_j) = \frac{\ln q(x_i|x_j|\Omega)}{\ln q(x_i|\Omega)q(x_j|\Omega)}$$

In order to reduce the set of solutions without reducing it to a single solution, it would be natural to consider the simple class of solutions where one of $\overline{a}(x_i|x_j)$ and $\overline{a}(x_j|x_i)$ is set to 0. Formally, for any $i \neq j$, an $ij$-directed solution is one where $\overline{a}(x_i|x_j) = 0$ and

$$\overline{a}(x_j|x_i) = \frac{\ln \frac{q(x_i|x_j|\Omega)}{q(x_i|\Omega)q(x_j|\Omega)}}{\ln q(x_i|\Omega)}.$$

Note that for a directed solution,

$$q(x_i|\Omega)^{\overline{a}(x_j|x_i)} = \frac{q(x_i,x_j|\Omega)}{q(x_i|\Omega)q(x_j|\Omega)},$$

where the expression on the right-hand side is a measure of statistical dependence.

6 Application

7 Related Literature

7.1 Psychology

The classic Heuristics and Biases program of Kahneman and Tversky (henceforth KT) is based on the idea that people’s judgements are based on heuristics. These heuristics can be useful in many situations, but they can also give rise to systematic biases relative to the standard Bayesian model. KT organize the evidence they accumulated under three heuristics: Representativeness, Availability, and Anchoring and Adjustment. See Tversky and
Kahneman (1974) for a review. We discuss the relationship between our model and each of these in turn:

The Anchoring and Adjustment heuristic describes the evidence that beliefs are responsive to possibly irrelevant anchors. Our model does not speak to that evidence.

The Availability heuristic organizes evidence that people judge as more likely those events for which they can readily recall more examples. KT emphasize salience as the determining feature. Salient examples and experiences may be responsible for stronger associations in our model, and in this sense, our model contains the spirit of the Availability heuristic. However, the notion of association is more general than that of salience: associations can be shaped by salience – for instance, a personal experience may create a stronger association than observing someone else’s experience – but they are also shaped by other factors such as the frequency of co-occurrence.

The Representativeness heuristic organizes evidence that people’s update reflects the representativeness of the information rather than the Bayesian update of prior beliefs. The heuristic dependence on representativeness of information gives rise to various biases, for instance, base-rate neglect, conservatism, the gambler’s fallacy, the hot-hand fallacy, the disjunction and conjunction fallacies. In formulating this heuristic, KT hypothesize that people’s updates mainly respond to similarity. In our model, updates respond to the more general notion of associations, and to the extent that similarity creates association, our model captures the spirit of Representativeness. Moreover, the model is flexible enough to capture the noted biases.

The use of the general notion of association rather than more specific notions like salience and similarity has the advantage for economics that it can calibrated using empirical frequencies, thereby opening the door to empirical analysis.

### 7.2 Cognitive Science

Artificial intelligence studies artificial neural networks with a focus on how to structure and train neural networks to improve their ability to predict. See also adaptive control theory (Andersen 1993). While this paper takes inspiration from this literature, the focus is very different. With an eye towards tractability for economic applications, we employ a very simple form of network, and hypothesize a simple rule for training the network by data.
7.3 Economics

Much of research on beliefs in economics generalize Bayes’ Rule in order to accommodate particular findings or satisfy particular properties. For instance, Rabin (2002) seeks to explain the law of small numbers, Epstein et al (2008) model updating that may be biased towards the prior or the data, and Gennaioli and Shleifer (2010) seek to model the Representativeness heuristic. The emphasis in this paper is on how beliefs are formed, and updating is a by-product of the model. The main structure in the model comes from how updated beliefs relate with each other through their reliance on an underlying associative network, and how this network may be shaped by experience.

[To be added: Cripps (2018), Ortoleva (2012), Zhao (2017)]

At least as relevant as the substantive literature on updating is the very narrow literature on belief formation. Spiegler (2016) models an agent in terms of a Bayesian network: a DAG (directed acyclic graph) is used to represent the agent’s presumed direction of causation between variables, and the agent’s beliefs are computed using a factorization formula and objective conditional probabilities. Intuitively, the agent has a subjective causal theory of the world, which guides her as she uses specific objective conditional probability from the data to construct her beliefs. To the extent that her subjective causal theory misses correlations that exist in the data, her subjective beliefs will differ from the objective probabilities. To compare with our model: The associative network we use are not DAGs. Each pair of nodes has two links (one in each direction) and moreover, the strength of these links matter. Spiegler’s agent uses the standard Bayesian network formula to translate her data and DAG into a belief, whereas data is encoded in some manner into our agent’s associative network and the resulting beliefs will generically not satisfy the Bayesian network factorization formula.\(^{10}\)

Subjective beliefs are updated by Bayes rule in Spiegler’s model whereas our agent’s updated beliefs are determined by the associative network. Finally, Spiegler’s agent believes in a uni-directional causal relationship between variables. In our model, the relationship is associative: the agent associates the occurrence of \(x_i\) after noting the occurrence of \(x_j\), and this can differ from how she associates the occurrence of \(x_j\) after noting the occurrence of \(x_i\).

The model of case-based inductive inference by Gilboa and Schmiedler (2003) considers an agent who ranks the likelihood of different eventualities.

\(^{10}\)Associations do not have the same mathematical properties as probabilities. For instance, \(a(x|z)\) does not equal \(a(x|y)a(y|z)\).
in a given problem. The model relates memory banks of past cases with a likelihood relation on eventualities. This differs from the Bayesian model conceptually in that the case-based agent looks back at the past in order to evaluate likelihoods, whereas the Bayesian agent explicitly thinks in terms of her view of the world as it would exist in the future. To compare with our model: Other than being very different in spirit, our model retains the forward-looking nature of Bayesian beliefs but founds these beliefs on associations, which are backward-looking in the sense of being determined by past experiences. Moreover, while the memory bank is a primitive of the case-based model, in our model the past experience is encoded in the agent’s subjective network. The case-based model also requires the modeler to specify a similarity function, while our model is based on associations. Finally, while the mechanical way in which beliefs are generated in our model makes it hard to interpret it in terms of deliberation, the case-based procedure can very readily be interpreted in terms of an agent generating beliefs through deliberation – without any other means to form beliefs about outcomes, she purposefully uses past data along with her assessment of similarity to help form likelihood assessments.

One property of association-based beliefs that sharply distinguishes our model from the above two models is as follows: Associations can be built if the agent is faced with copies of the data. A news story that is aired continuously on television can form an association without one even being conscious of it, thereby affecting beliefs. But such repetition should not have any impact on the agents modelled in the above noted papers.

8 Conclusion

In order for an agent to arrive at a rational conclusion, she needs to engage in deliberation. However, the real world is sufficiently complex and cognitive/time resources sufficiently limited that deliberation will generically not be carried out to its conclusion. We hypothesize that after an agent has deliberated to the best of her abilities, she may let her intuition determine her final choice. In this paper we articulate a way to conceptualize intuition and to model it mathematically in a manner that makes the model tractable and amenable to empirical application.

There are several directions for future work. A characterization of intuitive beliefs in terms of properties of beliefs would supply key tests for the
Another important direction is to study how an agent’s associative network may be shaped by the data. In this paper we hypothesized that the intuitive process can correctly learn the likelihood of simple events. One can theorize further, however. If an agent is already holding an asset, she may be more sensitive to parts of the data that speak to the possibility of making large gains or losses, and this may influence subsequent behavior. That is, the agent’s utility function may also enter a theory that links associative networks with data. With a theory of network formation in place, one can analyze how markets may react to changes in fundamentals.

We close with two comments on the role of intuition in rationality. The first is that if intuition is rooted in data, it should be viewed as a informative, albeit noisy, signal. Consequently, it should be considered rational to recommend an agent to follow their intuition after they have gone as far as their deliberation will take them. The second comment is based on the observation that deliberation is little more than the “cleaning up” of intuition. That is, deliberation is an operation on the material supplied by intuition. If deliberative conclusions are constructed from intuition, then properties of intuition may restrict where an agent’s deliberation will take them. In particular, deliberation may not guarantee the attainment of rationality.

A Appendix
To be added.

References


