Monotonicity of Optimal Flow Control for Failure Prone Production Systems[1]

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Abstrct. In this paper, we consider the problem of optimal flow control for a production system with one machine which is subject to failures and produces one part type. In most previous work, it has been assumed that the machine has exponential up and down times (i.e., its state process is a Markov process). The system considered in our study has *general* machine up and down times. Our main result is establishing monotone properties for the optimal control policy.

Key Words. Failure prone production systems, optimal flow control, hazard rate function, monotonicity.

1 Introduction

Most previous work on the problem of optimal flow control for failure prone production systems has been focused on systems whose underlying state processes are Markov processes. Olsder and Suri (Ref. 1) were among the first to study these systems and modeled them as systems with *jump Markov disturbances* based on Rishel's formalism (Ref. 2). Kimemia and Gershwin (Ref. 3), Akella and Kumar (Ref. 4), and Bielecki and Kumar (Ref. 5) showed that for systems with *homogeneous* Markov state processes (i.e., whose transition rates are constant) the so-called hedging point policy is optimal control, based on which several numerical approaches have then been proposed to find optimal or near-optimal controls (e.g., see Refs. 6-9).

The problem of finding optimal flow control becomes much more difficult when the state processes of the systems are not homogeneous Markov processes (i.e., when their transition rates are not constant and may depend on age and/or production rate). Boukas and Haurie (Ref. 10) considered a system which has two machines with age dependent failure rates and where preventive maintenance is a decision option. Sethi et al. (Ref. 11), Hu and Xiang (Ref. 12), and Liberopoulos (Ref. 13) studied systems with multiple states. Systems with machine failure rates depending on production rates were considered by Hu, Vakili, and Yu (Ref. 14) and Liberopoulos and Caramanis (Ref. 15). Hu and Xiang (Ref. 16) studied the one-machine and one-part-type system with general up and down times. They showed that the system under the hedging point policy is "equivalent" to a $GI/G/1$ queue; therefore, the existing results in queueing theory can be applied to obtain the steady-state probability distribution function of the surplus process under the hedging point policy. Systems with deterministic machine up times were considered in Hu and Xiang (Ref. 17). Tu, Song and Lou (Ref. 18) studied a so-called preventive hedging point policy for systems with general up and down times.

In this paper, we consider the one-machine and one-part-type system with general (i.i.d.) machine up and down times. Our work reported here is mainly motivated by our previous work in Ref. 12 and by the numerical results observed by Liberopoulos (Ref. 13). One result we proved in Ref. 12 is that if the machine up time has Erlang distribution, then the hedging point of the optimal control at each stage of the Erlang distribution monotonically increases as the stage number of the Erlang distribution increases (Theorem 4.2 of Ref. 12). However, as pointed out in Ref. 12, the stages of the Erlang machine up time are merely mathematical states and are not observable. It would be more interesting if we could establish the monotone property for the optimal control with respect to the age of the machine up time since it is the physical state we can observe. In fact, this type of monotonicity with respect to the age of the machine up time has been observed by Liberopoulos (Ref. 13) numerically. In this paper, we show that the optimal control has the same monotone property as that of the hazard rate of the machine up time, i.e., if the hazard rate is an increasing (respectively, decreasing) function with respect to the age, then the inventory level we need to maintain to hedge against future shortages brought by machine failures should also be increasing (respectively, decreasing) with respect to the age of the machine up time.

We should point out that in general it is extremely difficult to analytically obtain the optimal control even for some very simple systems with Markov state processes. Therefore, one often has to resort to numerical solutions or near-optimal control design, in which case knowledge of structure properties of the optimal control such as monotonicity can be very helpful. They can be used to assist the derivation of approximate procedures to design reasonable near-optimal controls and in some cases simplify the problem and obtain the optimal control analytically.

The rest of this paper is organized as follows. In Section 2, we first formulate the problem of optimal flow control for the system with one machine and one part type. We show that its optimal control belongs a class of switching curve policies. We then illustrate how a system with general machine up time can be approximated by a system with Coxian machine up time. The system with Coxian machine up time is discussed in Section 3. Our main result is presented and proved in Section 4. Section 5 contains a proof for Lemma 2.2. Finally, Section 6 is a conclusion.

2 Problem Formulation

The system we consider has one machine and produces one part type. The system tries to meet a constant demand rate d and backlog is allowed. The machine has two states: up and down, which are denoted by 0 and 1, respectively. When the machine is up, it can produce at a maximum rate r. Denote the production surplus at time t by $x(t)$; a positive value of $x(t)$ represents inventory while a negative value represents backlog. Let $\alpha(t) \in \{0,1\}$ be the state of the machine at time t, $a(t)$ be the age of the machine up time at t (for notational simplicity, we let $a(t)$ denote the age of the machine down time for those time intervals during which the machine is down), and $u(t)$ be the controlled production rate of the machine at t under a control policy π. $x(t)$ can then be characterized by the following differential equation

$$\frac{dx(t)}{dt} = u(t) - d, \tag{1}$$

where $0 \leq u(t) \leq \alpha(t)r$.

Given a control policy π, we are interested in the following expected discounted cost associated with it

$$J_\alpha^\pi(x,a) = E\left[\int_0^\infty g(x(t))\exp(-\gamma t)dt \middle| x(0)=x, \alpha(0)=\alpha, a(t)=a\right], \tag{2}$$

where $\gamma > 0$ is the discounted rate and $g(\cdot)$ is a strictly convex function minimized at g^* (in most cases, $g^* = 0$). The function penalizes the controller for failing to meet demand and for keeping an inventory of parts. Throughout this paper, we shall always assume that $J_\alpha^\pi(x,a)$ exists under any initial condition (α, x). Our goal is to find an optimal control which minimizes the expected discounted cost.

Suppose the machine up time has cumulative distribution function $F(\cdot)$ and density function $f(\cdot)$. We define its hazard rate function by

$$\lambda(a) = \frac{f(a)}{1 - F(a)}.$$

Let $J_\alpha(x,a)$ denote the minimum expected discounted cost, i.e.,

$$J_\alpha(x,a) = \min_\pi J_\alpha^\pi(x,a).$$

At those points where $J_1(x,a)$ is differentiable, the following well-known Hamilton-Jacobi-Bellman (HJB) equation holds,

$$\min_{0 \le u \le r} (u-d)\frac{\partial J_1(x,a)}{\partial x} + \frac{\partial J_1(x,a)}{\partial a} = -g(x) + (\gamma + \lambda(a))J_1(x,a) - \lambda(a)J_0(x,0). \qquad (3)$$

Lemma 2.1 For every fixed a, $J_1(x,a)$ is strictly convex with respect to $x \in \Re$.

The proof of Lemma 2.1 is similar to that of Theorem 5.1 in Ref. 19 (also see Ref. 4), hence we shall not repeat it here. Since for every fixed a $J_1(x,a)$ is a strictly convex function, it has a unique minimum point $z(a)$ and is differentiable on $\Re$ except on a countable set, and furthermore whenever its derivative exists we have

$$\frac{\partial J_1(x,a)}{\partial x} \begin{cases} > 0 & \text{for } x > z(a) \\ < 0 & \text{for } x < z(a) \end{cases} \qquad (4)$$

We also know that

$$\frac{\partial J_1(x,a)}{\partial x} \text{ is a strictly increasing function of } x \text{ for every fixed } a. \qquad (5)$$

Based on (3) and (4) it is clear that the optimal control π^* must satisfy

$$u^*(x,a) = \begin{cases} r & \text{if } x < z(a); \\ 0 & \text{if } x > z(a). \end{cases} \qquad (6)$$

We call the control policy defined by (6) switching curve policy and $z(a)$ switching curve. When $z(a) \equiv z$ (a constant), the switching curve policy becomes the so-called hedging point policy, and z is the hedging point.

In this paper, we are interested in properties of the switching curve $z(a)$. We shall show that $z(a)$ has the same monotone property as that of the hazard rate function $\lambda(a)$, i.e., if $\lambda(a)$ is monotonically increasing (respectively, decreasing), so is $z(a)$. Intuitively, this is quite simple: if $\lambda(a)$ is an increasing (respectively, decreasing) function, then the greater the age of the machine up time, the more (respectively, less) likely the machine will fail, hence the higher (respectively, lower) the inventory level we need to maintain to hedge against future shortages brought by machine failures, which in turn implies that $z(a)$ is also an increasing (respectively, decreasing) function. As a special case of this result, if $\lambda(a)$ is constant, so is $z(a)$, i.e., if the machine up time is exponentially distributed, then the hedging point policy is optimal. Finally, we should mention that for the case in which the machine up time is deterministic ($\lambda(a)$ is increasing), the switching curve $z(a)$ is explicitly obtained by Hu and Xiang (Ref. 17), which is an increasing piece-wise linear function.

To obtain the above monotone property, we will take an indirect approach. Instead of considering a general machine up time, we focus on the system with Coxian machine up time.

Definition 2.1 The class of *Coxian distributions* has representation as in Figure 1. That is there are n exponentials with rates $\mu_n, \ldots, \mu_1$ in series, but having passed stage k, it leaves with probability p_k (thus $p_1 = 1$). We call p_k failure probability of the Coxian distribution at stage k. With $q_k = 1 - p_k$, the Laplace transform of the Coxian distribution is given by

$$F_n^*(s) = \sum_{k=1}^{n-1} q_n q_{n-1} \cdots q_{k+1} p_k \prod_{i=k}^{n} \frac{\mu_i}{s + \mu_i} + p_n \frac{\mu_n}{s + \mu_n} \tag{7}$$

figure=fig2.ps,width=5.5in,height=0.7in

Figure 1: Coxian Distribution

Remark 2.1 Equivalently, the Coxian distribution can also be represented by the first passage time from state n to 0 of a finite-state Markov chain with $n+1$ states $n, \ldots, 1, 0$, and transition rates $\lambda_{k0} = \mu_k p_k$, $\lambda_{k,k-1} = \mu_k q_k$ $(k = n, \ldots, 1)$, and all other λ_{ij}'s are equal to zero. Hence state 0 of the Markov chain is an absorbing state. We call λ_{k0} the failure rate at state k. Clearly, if $\mu_n = \cdots = \mu_1$, then the failure rates are proportional to the failure probabilities.

It is well-known that the class of Coxian distributions is dense in the set of all probability distributions on $(0, \infty)$, e.g., see Asmussen (Ref. 20, p.76). However, we need a stronger result given in the following lemma, which relates the failure probabilities (rates) of a Coxian distribution to the hazard rate of the distribution function approximated by the Coxian distribution.

Lemma 2.2 Any probability distribution $F(x)$ on $(0, \infty)$ with bounded density function $f(x)$ can be approximated arbitrarily closely (in the sense of weak convergence) by a Coxian distribution such that:

1) the stage number of the Coxian distribution corresponds to the age of the probability distribution $F(x)$ (the closer to stage 0, the greater the age), and

2) the failure probabilities (rates) of the Coxian distribution are proportional to the corresponding hazard rates of the probability distribution $F(x)$.

In particular, if the hazard rate is increasing (respectively, decreasing) function, then the failure probability (rate) of the corresponding Coxian approximation increases (respectively, decreases) as the stage number decreases.

figure=fig1b.ps

Figure 2: Coxian Machine Up Time

The proof of Lemma 2.2 will be given in Section 5. Based on Lemma 2.2, we can then replace a general machine up time by a Coxian machine up time. As we shall see in the next section that for the system with Coxian machine up time the optimal control is a hedging point policy with one hedging point for each stage of the Coxian distribution. Therefore, monotonicity of the hedging points with respect to the stage number in the Coxian distribution implies monotonicity of $z(a)$ with respect to the age of the original machine up time. So in the next two sections, we will focus our discussions on the case in which the machine up time has Coxian distribution.

3 The System with Coxian Machine Up Time

We now consider the system with Coxian machine up time. Suppose the machine up time has n stages, i.e., the machine has $n+1$ states $\{0, 1, \ldots, n\}$, where state 0 corresponds to the complete failure state (with zero capacity). Clearly, the machine state process is now a Markov process with transition rate λ_{ij}: $\lambda_{ij} = 0$ if (i) $i < j$ unless $i = 0$ and $j = n$, or (ii) $i - j > 1$ unless $j = 0$. The transition diagram of the Markov chain is given in Figure 2. We note λ_{0n} is the hazard rate of the machine down time which in general depends on the age of the machine down time.

We shall use the same notation as defined in Section 2, but note we now use $\alpha(t) \in \{0, 1, \ldots, n\}$. Since now the machine state process is a Markov chain, the age of the machine up time is no longer needed (which is in fact replaced by the stages of the Coxian distribution). Therefore, the expected discounted cost associated with a control policy π defined by (2) for general machine up time should be modified as

$$J_\alpha^\pi(x) = E\left[\int_0^\infty g(x(t))\exp(-\gamma t)dt \middle| x(0)=x, \alpha(0)=\alpha\right], \tag{8}$$

for $\alpha = 0, 1, \ldots, n$. The minimum expected discounted cost is then defined by

$$J_\alpha(x) = \min_\pi J_\alpha^\pi(x),$$

which satisfies the following HJB equation

$$\begin{aligned} \min_{0\le u_1\le r}(u_1 - d)\frac{dJ_1(x)}{dx} &= -g(x) + (\lambda_{10}+\gamma)J_1(x) - \lambda_{10}J_0(x) \\ \min_{0\le u_\alpha\le r}(u_\alpha - d)\frac{dJ_\alpha(x)}{dx} &= -g(x) + (\lambda_{\alpha 0}+\lambda_{\alpha(\alpha-1)}+\gamma)J_\alpha(x) - \lambda_{\alpha 0}J_0(x) - \lambda_{\alpha(\alpha-1)}J_{\alpha-1}(x), \end{aligned} \tag{9}$$

for $\alpha = 2, \ldots, n$. The HJB equation (9) can be viewed as a discrete version of the HJB equation (3). Similarly to Lemma 2.1, we now have

Lemma 3.1 For each α, $J_\alpha(x)$ is a strictly convex function.

The proof of Lemma 3.1 is essentially the same as that of Lemma 2.1. Let z_α denote the unique minimum of $J_\alpha(x)$. Since $J_\alpha(x)$ is strictly convex, we have

$$\frac{dJ_\alpha(x)}{dx}\begin{cases} < 0, & \text{for } x < z_\alpha; \\ > 0, & \text{for } x > z_\alpha, \end{cases} \tag{10}$$

and

$$\frac{dJ_\alpha(x)}{dx} \text{ is a strictly increasing function.} \tag{11}$$

We now point out that in deriving the HJB equation (9) the differentiability of $J_\alpha(x)$ is required. However, we can in fact replace $dJ_\alpha(x)/dx$ by the right or left derivative of $J_\alpha(x)$. Similar to (6), we now have the following condition which the optimal control π^* must satisfy

$$u_\alpha^*(x) = \begin{cases} r, & \text{for } x < z_\alpha; \\ 0, & \text{for } x > z_\alpha. \end{cases} \tag{12}$$

Thus the optimal control policy is a hedging point policy with one hedging point for each machine state. Before closing this section, we present the following lemma which will be useful for us to establish the monotonicity in the next section.

Lemma 3.2 If $r > d$, then $(u_\alpha^*(x) - d)dJ_\alpha(x)/dx$ is decreasing with respect to x for $x \leq z_\alpha$ and increasing for $x \geq z_\alpha$.

Lemma 3.2 follows immediately from the strict convexity of $J_\alpha(x)$ and (12).

4 Main Result

To obtain our main result (the monotone property), we first need the following result:

Lemma 4.1 $z_\alpha \geq g^*$, for $\alpha = 1, \ldots, n$. Recall g^* is the minimum point of $g(x)$.

The proof of Lemma 4.1 is very much similar to that of Lemma 2.2 in Ref. 12 (also see Section 3 of Ref. 4). We now present our main result in the following theorem.

Theorem 4.1 For the system with Coxian machine up time, which is described in Section 3, if $\lambda_{n0} \leq \lambda_{(n-1)0} \leq \cdots \leq \lambda_{10}$ and $r \geq d$, then $z^* \leq z_n \leq z_{n-1} \leq \cdots \leq z_2 \leq z_1$, i.e., the value of the hedging point $\lambda_{\alpha 0}$ at each stage α of the Coxian machine up time increases as the stage number α decreases.

Proof. We use induction. First, we need to prove that $z_2 \leq z_1$. Based on the HJB equation (9) we have

$$(u_1^*(x) - d)\frac{dJ_1(x)}{dx} = -g(x) + (\lambda_{10} + \gamma)J_1(x) - \lambda_{10}J_0(x) \quad (13)$$

$$(u_2^*(x) - d)\frac{dJ_2(x)}{dx} = -g(x) + (\lambda_{20} + \lambda_{21} + \gamma)J_2(x) - \lambda_{20}J_0(x) - \lambda_{21}J_1(x) \quad (14)$$

Combining (13) and (14) by canceling $J_0(x)$, we obtain

$$\begin{aligned}
& (u_2^*(x)-d)\frac{dJ_2(x)}{dx} - (\lambda_{20}+\lambda_{21}+\gamma)J_2(x) \\
= \quad & -(1-\frac{\lambda_{20}}{\lambda_{10}})g(x) - \lambda_{21}J_1(x) + \frac{\lambda_{20}}{\lambda_{10}}\left[(u_1^*(x)-d)\frac{dJ_1(x)}{dx} - (\lambda_{10}+\gamma)J_1(x)\right]. \quad (15)
\end{aligned}$$

Since $z_1 \geq g^*$, both $g(x)$ and $J_1(x)$ are strictly increasing for $x \geq z_1$. This together with Lemma 3.2 tells us that the right-hand side of (15) is strictly decreasing for $x \geq z_1$. On the other hand, the left-hand side of (15) is strictly increasing for $x \leq z_2$. Therefore, we have $z_2 \leq z_1$.

We now assume that

$$g^* \leq z_{\alpha-1} \leq \cdots \leq z_2 \leq z_1 \qquad (16)$$

We now want to prove that $z_\alpha \leq z_{\alpha-1}$. Again, we use induction. We first prove $z_\alpha \leq z_1$. This is in fact similar to our proof for case $z_2 \leq z_1$. Again, the HJB equation (9) gives us

$$(u_1^*(x)-d)\frac{dJ_1(x)}{dx} = -g(x) + (\lambda_{10}+\gamma)J_1(x) - \lambda_{10}J_0(x) \qquad (17)$$

$$(u_\alpha^*(x)-d)\frac{dJ_\alpha(x)}{dx} = -g(x) + (\lambda_{\alpha 0}+\lambda_{\alpha(\alpha-1)}+\gamma)J_\alpha(x) - \lambda_{\alpha 0}J_0(x) - \lambda_{\alpha(\alpha-1)}J_{\alpha-1}(x) \qquad (18)$$

Similar to (15), we now have

$$\begin{aligned}
& (u_\alpha^*(x)-d)\frac{dJ_\alpha(x)}{dx} - (\lambda_{\alpha 0}+\lambda_{\alpha(\alpha-1)}+\gamma)J_\alpha(x) \\
= \quad & -(1-\frac{\lambda_{\alpha 0}}{\lambda_{10}})g(x) - \lambda_{\alpha(\alpha-1)}J_{\alpha-1}(x) + \frac{\lambda_{\alpha 0}}{\lambda_{10}}\left[(u_1^*(x)-d)\frac{dJ_1(x)}{dx} - (\lambda_{10}+\gamma)J_1(x)\right].
\end{aligned} \qquad (19)$$

Based on our hypothesis (16) we have $z_1 \geq z_{\alpha-1} \geq g^*$, hence based on Lemma 3.2 and the fact that $g(x)$, $J_1(x)$, and $J_{\alpha-1}(x)$ are all strictly convex functions with minimum points g^*, z_1 and $z_{\alpha-1}$, respectively, we know that the right-hand side of (19) is strictly decreasing for $x \geq z_1$. On the other hand, the left-hand side of (19) is strictly increasing for $x \leq z_\alpha$. Therefore, we have $z_\alpha \leq z_1$. now let us suppose

$$z_\alpha \leq z_{k-1} \leq \cdots \leq z_1, \tag{20}$$

for some $2 \leq k \leq \alpha - 1$. We need to show $z_\alpha \leq z_k$. Based on hypothesis (16) $z_k \leq z_{k-1}$, so we have the following two cases to consider:

Case 1: $z_k = z_{k-1}$. It is trivial that $z_\alpha \leq z_k$ based on our second hypothesis (20).

Case 2: $z_k < z_{k-1}$. Again, based on the HJB equation we have

$$(u_k^*(x) - d)\frac{dJ_k(x)}{dx} = -g(x) + (\lambda_{k0} + \lambda_{k(k-1)} + \gamma)J_k(x) - \lambda_{k0}J_0(x) - \lambda_{k(k-1)}J_{k-1}(x) \tag{21}$$

Combining (21) and (18) we have

$$\begin{aligned}
& (u_\alpha^*(x) - d)\frac{dJ_\alpha(x)}{dx} - (\lambda_{\alpha 0} + \lambda_{\alpha(\alpha-1)} + \gamma)J_\alpha(x) \\
= \; & -(1 - \frac{\lambda_{\alpha 0}}{\lambda_{k0}})g(x) - \lambda_{\alpha(\alpha-1)}J_{\alpha-1}(x) + \frac{\lambda_{\alpha 0}}{\lambda_{k0}}\lambda_{k(k-1)}J_{k-1}(x) \\
+ \; & \frac{\lambda_{\alpha 0}}{\lambda_{k0}}\left[(u_k^*(x) - d)\frac{dJ_k(x)}{dx} - (\lambda_{k0} + \lambda_{k(k-1)} + \gamma)J_k(x)\right]
\end{aligned} \tag{22}$$

Since $z_{\alpha-1} \leq z_k < z_{k-1}$, similar to (15) and (19) we can show that the right-hand side of (22) is strictly decreasing for $x \in [z_k, z_{k-1}]$. On the other hand, the left-hand side of (22) is strictly increasing for $x \leq z_\alpha$. Therefore $z_\alpha \leq z_k$. This completes our proof. □

Similar to Theorem 4.1, we also have

Theorem 4.2 For the system with Coxian machine up time, which is described in Section 3, if $\lambda_{n0} \geq \lambda_{(n-1)0} \geq \cdots \geq \lambda_{10}$ and $r \geq d$, then $g^* \leq z_1 \leq z_2 \leq \cdots \leq z_n$, i.e., the value of the hedging point $\lambda_{\alpha 0}$ at each stage α of the Coxian machine up time increases as the stage number α increases.

Its proof is essentially similar to the proof of Theorem 4.1, hence we omit it here.

Remark 4.1 Both Theorems 4.1 and 4.2 can be easily generalized to the following two cases i) maximum production rates at different stages have different values (which are not less than the demand rate d), and ii) $\lambda_{0\alpha} \neq 0$ for all $\alpha = 1, 2, \ldots, n$.

Remark 4.2 Based on Lemma 2.2, we can approximate a general machine up time arbitrarily closely by a Coxian machine up time. Furthermore, in such an approximation, the stage number of the Coxian distribution corresponds to the age of the original machine up time (the smaller the stage number the greater the age), and the failure rates are proportional to their corresponding hazard rates. Therefore, if the hazard rate function of the original machine up time is an increasing (respectively, decreasing) function with respect to its age, then its corresponding Coxian approximation has the failure rate increasing (respectively, decreasing) as the stage number decreases. Based on this relationship and Theorem 4.1 (respectively, Theorem 4.2) we know the switching curve of the optimal control policy is an increasing (respectively, decreasing) function with respect to the age of the machine up time if the hazard rate function is increasing (respectively, decreasing) with respect to the age.

5 Proof of Lemma 2.2

We now prove Lemma 2.2. Suppose $f(x)$ has support $(0, b)$ ($b \leq \infty$) i.e., $F(x) < 1$ for $x \in (0, b)$ and $F(b) = 1$. Hence $\lambda(x)$ is bounded on $(0, T)$, where $0 < T < b$. We define

$$\lambda_T(x) = \begin{cases} \lambda(x), & \text{for } x \in (0, T) \\ \epsilon_T, & \text{for } x \geq T, \end{cases}$$

where ϵ_T is a constant such that $\epsilon_T \to 0$ as $T \to b$. It is clear that $\lambda_T(x)$ is bounded on $(0, \infty)$ and the Laplace transform of the density function with hazard rate $\lambda_T(x)$ is given by

$$F_T^*(s) = \int_0^T \exp(-sx)\lambda(x) \exp\left(-\int_0^x \lambda(t)dt\right) dx + \frac{\epsilon_T}{s + \epsilon_T} \exp(-(s + \epsilon_T)T),$$

which converges to

$$F^*(s) = \int_0^b \exp(-sx)f(x)dx = \int_0^b \exp(-sx)\lambda(x)\exp\left(-\int_0^x \lambda(t)dt\right)dx$$

as $T \to b$. That is to say the distribution function $F_T(x)$ with hazard rate $\lambda_T(x)$ converges weakly to $F(x)$ as $T \to b$.

Based on the above discussions, it suffices for us to show that Lemma 2.2 holds for distribution function $F_T(x)$. We construct a sequence of Coxian distributions $\{F_n(x)\}$ as follows: $F_n(x)$ has n stages in which $\mu_n = \cdots = \mu_1 = nK/T$ and $p_k = (T/(nK))\lambda((n-k)T/(nK))$ for $k = 2, \ldots, n$ and $p_1 = 1$, where $0 < T < b$ and K is an upper bound of $\lambda_T(x)$ such that $p_k \le 1$. Then the Laplace transform of the density function of $F_n(x)$ is given by

$$\begin{aligned} F_n^*(s) &= \frac{T}{nK}\sum_{k=0}^{n-2}\lambda\left(\frac{kT}{nK}\right)\left[\prod_{i=0}^{k-1}\left(1-\frac{T}{nK}\lambda\left(\frac{iT}{nK}\right)\right)\right]\left(\frac{Kn}{sT+nK}\right)^{k+1} \\ &+ \left[\prod_{i=0}^{n-1}\left(1-\frac{T}{nK}\lambda\left(\frac{iT}{nK}\right)\right)\right]\left(\frac{Kn}{sT+nK}\right)^{n}. \end{aligned}$$

Since

$$\begin{aligned} &\prod_{i=0}^{k-1}\left(1-\frac{T}{nK}\lambda\left(\frac{iT}{nK}\right)\right) = \exp\left(\sum_{i=0}^{k-1}\log\left(1-\frac{T}{nK}\lambda\left(\frac{iT}{nK}\right)\right)\right) \\ &= \exp\left(-\sum_{i=0}^{k-1}\left(\frac{T}{nK}\lambda\left(\frac{iT}{nK}\right)+O\left(\frac{1}{n^2}\right)\right)\right) = \exp\left(-\frac{T}{nK}\sum_{i=0}^{k-1}\lambda\left(\frac{iT}{nK}\right)+O\left(\frac{1}{n}\right)\right) \\ &= \exp\left(-\int_0^{kT/(nK)}\lambda_T(t)dt + O\left(\frac{1}{n}\right)\right) = \exp\left(-\int_0^{kT/(nK)}\lambda_T(t)dt\right)+O\left(\frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{Kn}{sT+nK}\right)^{k+1}\exp\left(-(k+1)\log\left(1+\frac{sT}{nK}\right)\right) \\ &= \exp\left(-\frac{kT}{nK}s+O\left(\frac{1}{n}\right)\right) = \exp\left(-\frac{kT}{nK}s\right)+O\left(\frac{1}{n}\right) \end{aligned}$$

(note in the above derivations, all $O(1/n)'$s are uniform with respect to k,) we have

$$F_n^*(s) = \frac{T}{nK}\sum_{k=0}^{n-2}\lambda\left(\frac{kT}{nK}\right)\left(\exp\left(-\int_0^{kT/(nK)}\lambda_T(t)dt\right)+O\left(\frac{1}{n}\right)\right)\left(\exp\left(-\frac{kT}{nK}s\right)+O\left(\frac{1}{n}\right)\right)$$

$$
\begin{aligned}
& + \exp\left(-\int_0^{T/K} \lambda_T(t)dt\right) \exp(-Ts/K) + O\left(\frac{1}{n}\right) \\
= \quad & \frac{T}{nK} \sum_{k=0}^{n-2} \lambda\left(\frac{kT}{nK}\right) \exp\left(-\int_0^{kT/(nK)} \lambda_T(t)dt\right) \exp\left(-\frac{kT}{nK}s\right) \\
& + \exp\left(-\int_0^{T/K} \lambda_T(t)dt\right) \exp(-Ts/K) + O\left(\frac{1}{n}\right) \\
\to \quad & \int_0^{T/K} \exp(-sx)\lambda_T(x) \exp\left(-\int_0^x \lambda_T(t)dt\right) dx \\
& + \exp\left(-\int_0^{T/K} \lambda_T(t)dt\right) \exp(-Ts/K) \quad (n \to \infty) \\
= \quad & \int_0^{T/K} \exp(-sx)dF_T(x) + (1 - F_T(T/K)) \exp(-Ts/K) \\
\to \quad & \int_0^{\infty} \exp(-sx)dF(x) \quad (T \to b.)
\end{aligned}
$$

This completes our proof for Lemma 2.2. □

6 Conclusion

We have shown that the optimal control policy for the one-machine and one-part-type system with general machine up and down times has the same monotonicity as that of the hazard rate function of the machine up time. This type of structural property on the optimal control can be used to reduce the control space over which we have to search for the optimal control, and thus greatly facilitate the process of finding near-optimal controls. There are two possible directions we can pursue in future research to extend our work: 1) To extend the results of this paper to more complicated systems, for example, systems with multiple part types; and 2) To apply the monotone property established in this paper to find optimal or near-optimal controls.

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