Optimal Control for Systems with Deterministic Production Cycles

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Abstract—In this paper we consider a failure prone production system with deterministic production cycles. The objective is to find the optimal production rate to minimize long-run average cost. We first show that the optimal control policy belongs to a type of switching curve policy which has a special structural property and can be characterized by a single parameter. We then establish a relationship between the surplus process of the system under the optimal control policy and the workload process of a D/G/1 queue, based on which the existing results in queueing theory can be applied to obtain the steady-state probability distribution of the surplus process under the optimal control policy.

I. INTRODUCTION

Consider a production system which has a single machine and produces a single part-type. The system meets a constant demand rate \( d \) and backlog is allowed. The machine has two states: up and down, which are denoted by one and zero, respectively. The produced parts are represented by a fluid flow. When the machine is up, it can produce at any rate between zero and a maximum value \( r \). Denote the production surplus at time \( t \) by \( X_t \); a positive value of \( X_t \) represents inventory while a negative value represents backlog. Let \( \alpha_t \in [0,1] \) be the state of the machine at time \( t \) and \( u_t \) be the controlled production rate of the machine at time \( t \) under a control policy \( \pi \). \( X_t \) can then be characterized by the following differential equation

\[
\frac{dX_t}{dt} = u_t - d
\]

where \( 0 \leq u_t \leq \alpha_t r \). For each control policy \( \pi \), we use \( J_\pi \) to denote its long-run average expected cost

\[
J_\pi = \lim_{T \to \infty} \frac{1}{T} E \int_0^T (c^+ X_t^+ + c^- X_t^-) \, dt
\]

where \( X_t^+ = \max(X_t, 0) \), \( X_t^- = \max(-X_t, 0) \) and \( c^+ \), \( c^- \) are nonnegative constants. The goal is to find a control policy \( \pi^* \) to minimize \( J_\pi \).

A few variations of the above problem (e.g., systems with multiple machines and/or multiple part types) have been studied by many researchers (see [1], [2] and references therein). The key assumption used in almost all previous work, with a few exceptions [3]–[9], is that the machine state process \{\( \alpha_t \)\} is a Markov process, i.e., the machine up and down times are exponentially distributed. Under such an assumption, the system can be modeled as a system with jump Markov disturbances based on Risshel’s formulation ([3]), and it can be shown [10] that the optimal control is a hedging point policy. The Markov assumption, however, is not very realistic in many applications. So far, only limited work has been done for general systems.

In this paper we consider the one machine and one part-type system in which the machine has deterministic up time and general down time. A production system with deterministic up time, which is similar to ours, is also studied by Meyer et al. [11]. The system we consider here can be ideally used to model production systems in which production is only interrupted after (approximately) a fixed amount of time since it starts. For example, consider a production system in which preventive maintenance is performed after the system is in operation for a fixed amount of time. If the system rarely fails due to its scheduled preventive maintenance, then its up time is approximately deterministic. Another example is an inventory system which replenishes its inventory continuously (with no interruption) but only receives demands (orders) periodically. In such a system each period corresponds to one up time and an order (a random variable) is received at the end of each period corresponding to total demands accumulated during one down time. It is also worth pointing out that the periodic review system with limited capacity studied in inventory theory ([12]) is very similar to our system and their relationship is investigated in a recent paper by Fu and Hu [14]. As we shall see, the well-known hedging point policy is no longer optimal for our system. Our analysis shows, however, that its optimal control policy belongs to a class of so-called switching curve policies whose switching curves have a very simple structure and can be characterized by a single parameter. Furthermore, we show that the surplus process of the system operated under a switching curve policy is related to the workload process of a D/G/1 queue, a result similar to the one established in [15] for the one machine and one part-type system under a hedging point policy (also see [16]–[19] for similar results on the relationship of fluid models to the workload process of queueing systems). This enables us to use the existing results in queueing theory to obtain the steady-state distribution function of the surplus process for the class of parameterized switching curve policies to which the optimal control belongs. If \( \{X_t; \, t \geq 0\} \) has steady-state probability distribution function \( F(x) \) under a control policy \( \pi \), then the average expected cost \( J_\pi \).
can be computed as the expected cost with respect to $F(x)$, i.e.,
\[ J_s = \int_{-\infty}^{\infty} \left( C^+x^+ + C^-x^- \right) dF(x). \]  
\[ (3) \]

Therefore, with the steady-state probability we obtained we can finally convert the original optimal control problem into an optimization problem with a single parameter, which is much simpler and easier to solve.

The rest of this paper is organized as follows. In Section II, we first establish a structural property for the optimal control. Then, we discuss the switching curve policies and show that the optimal switching curve can be characterized by a single parameter. In Section III, we show how the surplus process of the system under a switching curve policy is related to the workload process of a $D/G/1$ queue. This relationship is used in Section IV to obtain the steady-state probability distribution of the surplus process for the system under the class of switching curve policies to which the optimal switching curve policy belongs. A simple example with exponential machine down times is provided in Section V.

II. THE OPTIMAL CONTROL POLICY

In this section, we first establish a structural property for the optimal control based on sample path analysis. Then we discuss a class of so-called switching curve policies. Using the structural property established, we can show that when restricting ourselves to the switching curve policies, the switching curve for the optimal control belongs to a class of switching curves which can be characterized by a single parameter. Intuitively, the structural property of the optimal control we are about to establish can be interpreted as follows: to minimize the surplus cost, one needs to maintain a nonnegative surplus at the end of each machine up period to anticipate future capacity shortages brought by machine failures while keeping the surplus level close to zero as much as possible. Most of our derivations are quite straightforward. We hope that the accompanying figures greatly aid in the understanding of the derivations.

Suppose $\pi^*$ is the optimal control policy. Given the machine state process $\{\alpha_t\}$, let $\{u^*_t\}$ be the optimal control process generated under the optimal control policy $\pi^*$, and let $\{X^*_t\}$ be the corresponding surplus process determined by (1) with $u_t$ being replaced by $u^*_t$.

**Lemma 1**: There exists an optimal control $\{u^*_t\}$ that satisfies the following condition
\[ u^*_t = \begin{cases} r & \text{if } X^*_t < 0, \alpha_t = 1; \\ d & \text{if } X^*_t = 0, \alpha_t = 1. \end{cases} \]  
\[ (4) \]

Condition (4) can be explained as follows: When the surplus is negative the production rate should be set at the maximum rate $r$ so that the surplus can be brought back to zero level as quickly as possible; when the surplus is at zero level we should keep it at a nonnegative level as long as we can. In other words, we should always prevent the surplus from being negative. Based on Lemma 1, we shall henceforth assume the optimal control we consider satisfies (4).

**Proof**: If $\{u^*_t\}$ does not satisfy (4), we can construct another optimal control based on $\{u^*_t\}$ which satisfies (4). Suppose $\alpha_t = 1$. Let us first consider the case $X^*_t < 0$. We denote the first hitting time to 0 from $X^*_0$ by
\[ t^*_0 = \inf \{t: X^*_t = 0, t > t_0\}. \]

By definition, we have $X^*_t < 0$ for $t \in [t_0, t^*_0)$. We construct
\[ u_t = \begin{cases} \alpha_t & \text{if } t \in [t_0, t^*_0) \text{ and } X_t < 0; \\ \alpha_t + 1 & \text{if } t \in [t_0, t^*_0) \text{ and } X_t = 0; \\ u^*_t & \text{otherwise.} \end{cases} \]  
\[ (5) \]

\[ u_t = \begin{cases} \alpha_t & \text{if } t \in [t_0, t^*_0); \\ \alpha_t + 1 & \text{if } t \in [t_0, t^*_0); \\ u^*_t & \text{otherwise.} \end{cases} \]

Therefore, we have the following three cases:
1) $X^*_t < 0$ and $X^*_t < 0$. In this case, $X^*_t < 0$ for all $t \in [t_0, t^*_0)$, which implies $u^*_t = r$ for all $t \in [t_0, t^*_0)$. 
2) $X^*_t < 0$ and $X^*_t \geq 0$. Let $t^*_0$ be the first hitting time to zero from $X^*_t$. It immediately follows that $u^*_t = r$ for $t \in [t_0, t^*_0)$. Also it is clear that $X^*_t \geq 0$ for $t \in [t_0, t^*_0)$. We now construct another control $\{u_t\}$ based on $\{u^*_t\}$ as follows
\[ u_t = \begin{cases} r & \text{if } t \in [t_0, t^*_0) \text{ for } t \in [t_0, t^*_0); \\ u^*_t & \text{otherwise.} \end{cases} \]

We can easily verify that the surplus process $\{X_t\}$ associated with the control $\{u_t\}$ satisfies
\[ X_t \leq X^*_t, \quad \text{if } t \in [t_0, t^*_0); \\ X_t = X^*_t, \quad \text{otherwise.} \]
where \( S: [0, D] \to \mathbb{R} \) is piecewise differentiable and its derivative \( S'(a) \) (whenever it exists) satisfies \(-d \leq S'(a) \leq r - d\). We call \( S(\cdot) \) the switching curve.

The following result follows immediately from the proof of Lemma 2.

**Lemma 3:** The optimal switching curve is given by

\[
S(a) = \begin{cases} 
0 & \text{if } 0 \leq a < D - D/z_1 (r - d), \\
\frac{z_1 (D - a)}{(r - d)} & \text{if } D - D/z_1 (r - d) \leq a \leq D. 
\end{cases}
\] (7)

In (7), \( z \) is the inventory level at which the surplus needs to be maintained at the end of the machine up period to hedge against future capacity shortages brought by machine failures. Therefore, it plays a similar role as the hedging point in a hedging point policy. There is a significant difference, however, between the hedging point policy and the control policy defined by (7). Under the hedging point policy, the machine produces at its maximum rate until the surplus reaches the hedging level (i.e., the surplus is brought to the hedging level as fast as it could be). Under (7) one only has to make sure that the surplus is brought to the level \( z \) at the end of each machine up period and before that the surplus level should be kept as close as to zero (therefore minimizing the surplus cost). Obviously, this is because the machine up time is now deterministic, and we know exactly when the machine will fail so that the machine only needs to produce parts at its maximum rate toward the end of each machine up period to guarantee that the surplus level will eventually reach the level \( z \) before the machine fails.

In the remainder of this paper, we restrict ourselves to the switching curve policies. We shall show how to obtain the steady-state probability distribution function of the surplus process for the system operated under the switching curve policy (7).

### III. Queueing Equivalence

In this section we study the surplus process of the system under a switching curve policy. We shall show that the surplus levels at instances when the machine is up and down are “equivalent” to the system and waiting times of jobs of a \( D/G/1 \) queue, a result which is similar to the one established in Hu and Xiang [15] for the one machine and one part-type system under the hedging point policy. This queueing equivalent result enables us to use the existing results for single-server queues to obtain the steady-state probability distribution for the surpluses at these instances, based on which we can then derive the steady-state probability distribution of the surplus process for the system under the switching curve policy (7) using the level crossing technique.

Denote \( S(D) \) for \( z \). For the sake of convenience, we assume \( X_0 = z \) and \( \alpha_0 = D \) (i.e., \( \alpha_0 = 1 \) and \( \alpha_0^+ = 0 \)). Denote by \( t_{du} \), the length of the nth down time and by \( t_{du} \), the length of the nth up time. Therefore, \( t_{du} \)'s are i.i.d. random variables, say, with probability distribution function \( G(x) \) and density function \( g(x) \), and \( t_{du} \)'s are equal to \( D \). Let \( T_{du} \) be the epoch at which the nth machine down time starts and \( T_{du} \) be the epoch at which the nth machine up time starts, i.e.,

\[
T_{du} = \sum_{i=1}^{n-1} (t_{d,i} + t_{u,i}) \quad \text{and} \quad T_{du} = T_{du} + t_{d,u}.
\] (8)

For simplicity, we denote

\[
X_{d,u} \triangleq X_{d,u} \quad \text{and} \quad X_{d,u} \triangleq X_{d,u}.
\]

Let \( Y_{d,u} \) and \( Y_{d,u} \) correspond to the waiting time and the system time of job \( n \) in a \( GI/G/1 \)
queue with service times \( \{d_{n,d}: n = 1, 2, \ldots \} \) and interarrival times \( \{s_{n,r-d}: n = 1, 2, \ldots \} \). Since \( s_{n,r-d} \) is equal to a constant \( D(r-d) \), the \( G1/G/1 \) queue is in fact a \( D/G/1 \) queue. Therefore, we can apply various methods developed for single server queues in the literature to obtain the steady-state probability distribution functions of \( Y_{d,n} \) and \( Y_{s,n} \) (hence \( X_{d,n} \) and \( X_{s,n} \)) (e.g., see [20]). In the following sections, we shall simply assume that the steady-state probability distributions of \( X_{d,n} \) and \( X_{s,n} \) are available.

IV. DISTRIBUTION FUNCTION

In this section, we consider the switching curve policy (7). We use the level crossing technique to derive the steady-state distribution function of the surplus process under (7). We should point out, however, that the technique can also be used to obtain the steady-state distribution function of the surplus process under a general switching curve policy though the derivation becomes a bit more involved.

Denote the steady-state distribution function of \( X_{d,n} \) by \( F_{X,n}(x) \), and its density function by \( f_{X,n}(x) \). Define

\[
D_{1}(x) = \text{the number of} \ x \text{-downcrossings of the process} \ {X_{1}} \text{during} \ [0,t];
\]

\[
U_{0}(x) = \text{the number of} \ x \text{-upcrossings of the process} \ {X_{1}} \text{during} \ [0,t];
\]

Notice that the accumulated time that the process \( \{X_{1}\} \) takes value between \([x, x+dx]\) during \([0,t]\) is equal to

\[
\frac{U_{0}(x)dx}{r-d} + \frac{D_{1}(x)dx}{d}.
\]

Therefore, the steady-state probability density function of \( \{X_{1}\} \) is given by

\[
f_{X}(x) = \lim_{t \to \infty} \frac{1}{t} \left( \frac{U_{0}(x)}{r-d} + \frac{D_{1}(x)}{d} \right) \quad \text{for} \ x \leq z \text{ and } x \neq 0.
\]

(The interested reader is referred to [16] for a rigorous derivation of (13).) Note that

\[
\lim_{t \to \infty} \frac{U_{0}(x)}{t} = \lim_{t \to \infty} \frac{D_{1}(x)}{t}
\]

it then follows from (13) that

\[
f_{X}(x) = \frac{r}{d(r-d)} \lim_{n \to \infty} \frac{U_{0}(x)}{n} = \frac{r}{d(r-d)} \lim_{n \to \infty} \frac{U_{0}(x)}{n} + \frac{D_{1}(x)}{n} \text{ for } x \leq z \text{ and } x \neq 0.
\]

(Recall \( T_{s,n+1} \) is the epoch at which the \((n+1)\)th machine up time initiates.) To calculate \( \lim_{n \to \infty} \frac{U_{0}(x)}{n} \), we consider an down and up cycle \([T_{s,n}, T_{s,n+1}] \). First, it is obvious that the process \( \{X_{1}\} \) can have at most one \( x \)-upcrossing during \([T_{s,n}, T_{s,n+1}] \) (in fact during \([T_{s,n}, T_{s,n+1}] \)). Define

\[
c(x) = \begin{cases} D - \frac{x-z}{r-d} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}
\]

(See Fig. 5.) Then \( \{X_{1}\} \) has one \( x \)-upcrossing during \([T_{s,n}, T_{s,n+1}] \) if and only if \( x > (r-d)D < X_{n,n} \leq x + dc(x) \). Therefore

\[
U_{0}(x) = \sum_{i=1}^{n} 1 \{x - (r-d)D < X_{n,n} \leq x + dc(x)\}.
\]

It follows that

\[
f_{X}(x) = \frac{r}{d(r-d)} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T_{s,n+1}}
\]

where \( b = \frac{r}{(r+D)(d-r)} \) and \( R \) is the average machine down time. Taking the normalization factor into consideration, we have

\[
F_{X}(0) = 1 - \int_{-\infty}^{\infty} f_{X}(x) \, dx.
\]

Let \( J_{1} \) be the run-long average expected cost associated with the switching curve policy (7), then

\[
J_{1} = \int_{-\infty}^{\infty} (C^{+}x^{+} + C^{-}x^{-})f_{X}(x) \, dx.
\]

Therefore, the optimal switching curve (i.e., the optimal value of \( z \)) can be obtained by solving the following optimization problem:

\[
\min_{z} J_{1}.
\]

V. AN EXAMPLE

In this section, we study a simple example in which the machine down time is exponentially distributed with mean \( R \) and rate \( \mu = 1/R \). We will use the results obtained in the previous section to calculate \( f_{X}(x) \) and \( J_{1} \). Since now the down time is exponentially distributed, the corresponding \( D/G/1 \) queue becomes a \( D/M/1 \) with the mean interarrival time \( (r-d)D \) and the mean service time \( R/d \).

Hence we have (20)

\[
F_{X,n}(x) = e^{\frac{-(r-d)x}{R}} \left( \frac{x}{d} \right)^{n}, \quad \text{for } x \leq z,
\]

where \( \sigma \) is the unique solution to the following equation in the range \( 0 \leq \sigma < 1 \)

\[
\sigma = e^{\frac{(r-d)}{R}} \frac{z}{d}.
\]

Based on (15) we obtain

\[
F_{X}(x) = \int_{0}^{x} \left( 1 - e^{\frac{-(r-d)x}{R}} \left( \frac{x}{d} \right) \right) \, dx
\]

for \( x \leq z \) and \( z \neq 0 \).

When \( z < (r-d)D/r \), (13) gives

\[
f_{X}(x) = \begin{cases} b \left( 1 - e^{\frac{-(r-d)x}{R}} \left( \frac{x}{d} \right) \right) & \text{if } 0 < x \leq z, \\ be^{\frac{-(r-d)x}{R}} \left( \frac{x}{d} \right) & \text{if } x < 0. \end{cases}
\]
and

\[ J_i = \frac{C^i + b_z^2}{2} - C^i b_\phi \frac{\mu}{(1 - \mu)(r - d)D/d} \]
\[ \left[ \frac{\phi}{\mu^2(1 - \mu)^2} - \frac{d^2}{\mu^2(1 - \mu)^2} + \frac{d^2}{\mu^2(1 - \mu)^2} \right] \]
\[ + \frac{C^i - b_\phi \mu d}{\mu^2(1 - \mu)^2} e^{-\mu(1 - \mu)(r - d)D/d} \cdot \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right). \]

When \( d \geq d(r - d)D/r, \) (13) gives

\[ f(x) = \begin{cases} 1 - (1 - \mu)x + (r - d)D/r & \text{if } x \leq \frac{d}{r} \] \[ \frac{e^{-\mu(1 - \mu)(r - d)D/d}}{\mu(1 - \mu)} \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right) & \text{if } 0 < x \leq \frac{d}{r} - \frac{d}{r}D/r, \] \[ b_\phi \frac{(1 - \mu)x}{(1 - \mu)} \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right) & \text{if } x < 0. \] \]

and

\[ J_i = \frac{bC^i + d_z}{\mu(1 - \mu)} \]
\[ \left( \frac{r - d}{r} (D(1 - \mu) + 1) - e^{-\mu(1 - \mu)(r - d)D/d} \right) \]
\[ + \frac{bC^i + d^2_z}{\mu^2(1 - \mu)^2} \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right) \]
\[ + \frac{bC^i - d_z}{\mu^2(1 - \mu)} \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right) \]
\[ = \frac{C^i + d^2_z}{2\mu} \left( 1 - \frac{2}{\mu} + D \right) \]
\[ + \frac{bC^i + d^2_z}{\mu^2(1 - \mu)} \left( 1 - e^{-\mu(1 - \mu)(r - d)D/d} \right). \]

REFERENCES


