Structural Properties of Optimal Production Controllers in Failure-Prone Manufacturing Systems

Jian-Qiang Hu and Dong Xiang

Abstract— **A failure-prone manufacturing system with one part-type and multiple machine states is considered. Each machine state has a given production capacity but is associated with several possible failure stages. Although transition times between failure stages are assumed exponential, the existence of multiple stages within each machine state allows failure times to have a distribution more general than the exponential distribution. For such a system, it is shown that the well-known hedging point policy minimizes inventory and backlog costs for a constant demand rate. The value of the optimal hedging points is now a function of failure stage as well as machine state. By ordering machine states according to increasing distance from the zero capacity state and failure stages according to distance from the capacity reduction transition, we show that hedging points increase monotonically when the machine state number decreases or the stage number increases. This can be intuitively interpreted as follows: the closer to the zero capacity state the system is, the larger the hedging point should be. This structural property of the optimal control is very useful in searching for the optimal control or designing near-optimal controls.**

I. Introduction and Background

Recently, there has been a great deal of interest in the production rate control of manufacturing systems with machines subject to failure and repair. Perhaps the most important feature of these systems is that their dynamics are deterministic between machine failures and repairs. Olsder and Suri [13] were among the first to recognize this feature. They proposed to model these systems as systems with *jump Markov disturbances* based on Rishel's formalism [14]. In general, it is extremely difficult to obtain the true optimal controls for these systems since it requires to solve a complex Hamilton–Jacobi–Bellman (HJB) equation which can be only reduced to the solution of partial differential equations usually with unknown boundary conditions. Since we often have to resort to numerical solutions or near optimal controller designs, knowledge of the structure of the optimal control can be very useful. Structural properties of the optimal control can often assist the derivation of approximate procedures to design reasonable near-optimal controls and, in some cases, simplify the problem and obtain the optimal control analytically. This is exactly the motivation of the work reported in this paper.

The first structural result of the optimal production control of failure prone manufacturing systems was obtained by Kimemia and Gershwin [11]. They showed that the optimal control for such systems has a special structure called *hedging* point policy. In such a policy, a nonnegative production surplus is maintained when excess capacity is available to hedge against future shortages of parts brought about by machine failures. Their analysis is based on the well-known HJB equation for the optimal control and the optimal cost function. Based on the fact that the optimal control is a hedging point policy, Akella and Kumar [1] and Bielecki and Kumar [3] obtained the exact solution of the optimal control for a special case in which the system has a single machine and processes a single part-type. Their approach is to solve the system of differential equations which is obtained from the optimality conditions under a hedging point policy (e.g., the HJB equation for the optimal hedging point control policy). Sharifnia [15] later applied the same approach to the case of multiple machines and a single part-type, and Algoet [2] generalized it further to the multiple-machine and multiple-part-type case. They derived (partial) differential equations that characterize the steady-state probability density function under a given hedging point policy. Hu and Xiang [8] recently used a different approach to obtain the optimal control for the one-machine one-part-type system. They first established an equivalent relationship between the system under the hedging point policy and a single server queueing system, then used the existing results from queueing theory to obtain the steady-state probability density function. The first approximate procedure to find near-optimal control was developed by Gershwin, Akella and Choong [7]. They proposed to use a quadratic approximation for the optimal value function in the HJB equation. Caramanis and Sharifnia [6] use an approximate method to decompose a system with multiple part-types to several smaller systems with a single part-type, thus the problem becomes numerically tractable. Recently, Caramanis and Liberopoulos [5] applied the technique of perturbation analysis to design a near-optimal controller. These approximate procedures are based on the assumption that the desirable production rate control policies belong to the family of hedging point policies.

In this paper, we consider a system that produces a single part-type to meet a constant demand rate with the following characteristics: the system has several machine states, each machine state corresponds to a system failure mode (some of the machines failed) and has a specified production capacity, and the system makes a transition from one machine state to another when some machine breaks down or is repaired. We assume that the transition from one machine state to another forms a birth and death process. Intuitively, we can think of a machine state as the number of operational machines where a birth corresponds to a machine repair and a death to a machine failure. The system is discussed in detail in Section II, where some preliminary results are also presented. It is shown that for such a system the hedging point policy is still the optimal control.

In Section III, we first consider the case in which the birth and death rates at each machine state are constant. We show that the value of the optimal hedging point increases as the machine state (i.e., the number of operational machines) decreases. Our analysis is based on the HJB equation.[1]

In Section IV, we consider a more complicated system in which failure times are equal to the sum of several exponential random variables, i.e., failure times traverse several stages. Although at each stage the failure rate is constant, the failure rate at each machine state is not a constant any more. As a special case, failure times can have Erlang distributions. For such a system, the value of the optimal hedging point also depends on the failure stages within each machine state. We show that it increases as the stage number increases. Intuitively, our result can be interpreted as follows: the more likely the system will fail, the higher the hedging point should be. It is clear that our system is a special case in which failure rates depend on ages of machines (i.e., how long machines have been operational). In most previous studies, it is often assumed that failure rates are constant. In practice, the failure rate of a machine is usually not constant; it

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The authors are with Manufacturing Engineering Department, Boston University, Boston, MA 02215.

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[1] During the review process of this paper, it was brought to our attention that results similar to ours (Theorem 1) have also been independently obtained by Sethi *et al.* [16], [17]. The proof in [17], however, is very different from ours; furthermore, they assume that the production capacity decreases monotonically with respect to the machine state number while we do not.

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depends on many factors, such as the age of the machine. Usually, the older the machine, the more likely it will fail. Few studies have been conducted so far for systems with nonconstant failure rates. Boukas and Haurie [4] considered a two-machine system with age dependent failure rate. Numerical investigations can be found in Liberopoulos [12] where the structure of the optimal control for systems with nonconstant failure rates is tested. A related observation is that for one-machine one-part-type system with Erland failure times the value of the optimal hedging point increases as the stage number increases. This is a special case of the more general results established here.

Finally, in Section V, we provide a summary and point out some future research directions.

II. Preliminaries

The system under consideration is assumed to have $N+1$ machine states and produce a single part-type to satisfy a constant demand rate d with backlog allowed. We use $n \in \{0, 1, \cdots, N\}$ to denote the machine state of the system, where 0 corresponds to the state at which the system has failed completely and cannot produce any parts. At state n, the system can produce at a rate less than or equal to a specified maximum rate $\overline{u}_n$ ($\overline{u}_0 = 0$). Denote the machine state at time t by $n(t)$. We assume that $\{n(t): t \geq 0\}$ is a birth and death process with a birth rate μ_n and a death rate λ_n at state n ($n = 0, 1, \cdots, N$; $\mu_N = \lambda_0 = 0$). Denote the production surplus (positive or negative) at time t by $x(t)$. (A positive value of the surplus represents inventory while a negative value represents backlog.) $\{x(t): t \geq 0\}$ is a stochastic process characterized by the following differential equation

$$\frac{dx(t)}{dt} = u(t) - d \tag{1}$$

where $u(t)$ is the controlled production rate of the system at time t which satisfies the constraint $0 \leq u(t) \leq \overline{u}_{n(t)}$. We will use π to denote a control policy.

Given a control policy π, we are interested in the following expected discounted cost associated with it

$$J_n^\pi(x) = E\left[\int_0^\infty g(x(t))e^{-\gamma t}\, dt \middle| x(0) = x,\ n(0) = n\right] \tag{2}$$

where $\gamma > 0$ is the discounted rate and $g(\cdot)$ is a strictly convex function minimized at z^* (in most cases, $z^* = 0$). The function $g(x(t))$ penalizes the controller for failing to meet demand and for keeping an inventory of parts. Throughout this paper, we shall always assume that $J_n^\pi(x)$ exists under any initial condition (n, x). Our goal is to find an optimal control policy which minimizes the expected discounted cost.

let $J_n(x)$ denote the minimum expected discounted cost, i.e.,

$$J_n(x) = \min_\pi J_n^\pi(x). \tag{3}$$

At those points where $J_n(x)$ is differentiable, the following well-known HJB equation holds

$$\min_{0 \leq u_n \leq \overline{u}_n} (u_n - d)\frac{dJ_n(x)}{dx} = -g(x) + (\lambda_n + \mu_n + \gamma)J_n(x) - \lambda_n J_{n-1}(x) - \mu_n J_{n+1}(x) \tag{4}$$

for $n = 0, 1, \cdots, N$, where we define $J_{-1}(x) = J_{N+1}(x) \equiv 0$. Note that since $\overline{u}_0 = 0$, (4) becomes for $n = 0$

$$-d\frac{dJ_0(x)}{dx} = -g(x) + (\mu_0 + \gamma)J_0(x) - \mu_0 J_1(x). \tag{5}$$

Lemma 1: $J_n(x)$ is a strictly convex function of x.

The proof of Lemma 1 is the same as that of Theorem 5.1 in [18], and we shall not repeat it here. Since $J_n(x)$ is a strictly convex function, it has right and left derivatives, and furthermore, there exists z_n such that

$$\frac{d^+J_n(x)}{dx}, \frac{d^-J_n(x)}{dx} > 0 \quad \text{for } x > z_n \text{ and}$$
$$\frac{d^+J_n(x)}{dx}, \frac{d^-J_n(x)}{dx} < 0 \quad \text{for } x < z_n \tag{6}$$

where $+$ and $-$ denote the right and left derivatives, respectively. In fact, z_n is the unique minimum point of $J_n(x)$. We also now that both $\frac{d^+J_n(x)}{dx}$ and $\frac{d^-J_n(x)}{dx}$ are increasing functions and

$$\frac{d^-J_n(x)}{dx} \leq \frac{d^+J_n(x)}{dx}. \tag{7}$$

In deriving the HJB equation, the differentiability of $J_n(x)$ is required. We can in fact replace $dJ_n(x)/dx$ by $d^sJ_n(x)/dx$, where $s \triangleq \text{sign}(u_n - d)$. From (6) and (4), it is clear that the optimal control π^* must satisfy

$$u_n^*(x) = \begin{cases} \overline{u}_n & \text{if } x < z_n; \\ 0 & \text{if } x > z_n. \end{cases} \tag{8}$$

The control policy defined by (8) is often called hedging point policy and z_n is called hedging point. It is usually believed that $u_n^*(x) = \max(d, \overline{u}_n)$ at $x = z_n$. This is not crucial, however, to the rest of our analysis. In the next section, we shall investigate the order of the hedging points $z_1, z_2, \cdots, z_N$.

III. Main Result

We first present the following lemma.

Lemma 2: $z_n \geq z^*$, for $n = 1, \cdots, N$. Recall z^* is the minimum point of $g(x)$.

The basic idea of the proof given below is similar to the one given in Section III of [1] for the case $N = 1$, though it is somewhat more complicated.

Proof: Let $\{n(t): t \geq 0\}$ be the machine state process with $n(0) = n$, and let $\{x^*(t): t \geq 0\}$ and $\{u^*(t): t \geq 0\}$ be the corresponding surplus process and production rate process under the optimal control π^*, respectively. In what follows we construct a control policy π based on π^* such that $J_n^\pi(z^*) < J_n(z^*)$ if $z_n < z^*$ (recall $J_n(x) \triangleq J_n^{\pi^*}(x)$), which contradicts the fact that $J_n(x)$ is the minimum cost. Hence, $z_n \geq z^*$.

We consider the following control policy π:

$$u(t) = \begin{cases} \min(d, \overline{u}_{n(t)}) & \text{If } x(t) > x^*(t); \\ u^*(t) & \text{otherwise} \end{cases}$$

where $\{x(t): t \geq 0\}$ is the resulting surplus process corresponding to the production rate process $\{u(t): t \geq 0\}$ and the machine state process $\{n(t): t \geq 0\}$. We note that the production rate process $\{u(t): t \geq 0\}$ is defined as feedback on $x(t)$. Based on this construction, it is clear that if $x(t) = x^*(t)$ at $t = T$, then $x(t) = x^*(t)$ for all $t \geq T$ since $u(t) = u^*(t)$ for all $t \geq T$. We define $t_0 = \inf\{T: x(T) = x^*(T)\}$, and then $x(t) = x^*(t)$ for all $t \geq t_0$. Also, by the definition of t_0, we know that if $x(0) > x^*(0)$ then $x(t) > x^*(t)$ for all $0 \leq t < t_0$, which implies $u(t) \leq d$. Therefore, if $x(0) > x^*(0)$, we have $x(0) \geq x(t) > x^*(t)$ for all $0 \leq t < t_0$.

Suppose $z_n < z^*$. Let $x(0) = z^*$ and $x^*(0) = z_n$. It then follows from the above discussion that

$$x^*(t)\begin{cases} < x(t) \leq z^* & 0 \leq t < t_0; \\ = x(t) & t \geq t_0. \end{cases}$$

Therefore,

$$g(x^*(t))\begin{cases} > g(x(t)) & 0 \le t < t_0; \\ = g(x(t)) & t \ge t_0 \end{cases}$$

which implies $J_n^\pi(z^*) < J_n(z_n)$. On the other hand, since z_n is the minimum point of $J_n(x)$, we have $J_n(z_n) \le J_n(x)$ for all x, which leads to $J_n^\pi(z^*) < J_n(z^*)$. This completes the proof. Q.E.D.

We proceed next with the derivation of the main result.

Theorem 1: Supposing $\overline{u}_n \ge d$ $(n = 1, 2, \cdots, N)$, it follows that $z_1 \ge z_2 \ge \cdots \ge z_N \ge z^*$.

Proof: The theorem is proven if we show

i) $z_n \le z_{n-1}$;

ii) if $z_n > z^*$ then $J_n(x) - J_{n-1}(x)$ is increasing for $x \in [z_n - \Delta_n, z_n)$, where $\Delta_n > 0$, for $n = N, N-1, \cdots, 2$.

We use induction. First consider $n = N$. From the HJB equation (4) with $dJ_n(x)/dx$ being substituted by $d^s J_n(x)/dx$, we have

$$(u_N^*(x) - d)\frac{d^s J_N(x)}{dx} = - g(x) + (\lambda_N + \gamma)J_N(x) - \lambda_N J_{N-1}(x). \quad (9)$$

According to Lemma 2, we have $z_N \ge z^*$. If $z^* = z_N$, it follows from Lemma 2 that $z_{N-1} \ge z_N$. So we only need to consider the case $z^* < z_N$. Based on (7) and (8), we know that $(u_N^*(x) - d)d^s J_N(x)/dx$ is an increasing function for $x < z_N$. Note that since z_N and z^* are minimum points of $J_N(x)$ and $g(x)$, respectively, $-(\lambda_N + \gamma)J_N(x)$ and $g(x)$ are increasing for $x \in [z^*, z_N)$. It then follows from (9) that $J_{N-1}(x)$ is decreasing for $x \in [z^*, z_N)$, which implies that $z_N \le z_{N-1}$. Furthermore, we can rewrite (9) as

$$\lambda_N(J_N(x) - J_{N-1}(x)) = (u_N^*(x) - d)\frac{d^s J_N(x)}{dx} + g(x) - \gamma J_N(x)$$

from which we can conclude that $J_N(x) - J_{N-1}(x)$ is increasing for $x \in [z^*, z_N)$.

Suppose i) and ii) hold for $n = N, \cdots, K+1$. We now consider $n = K$. Same as for $n = N$, we only need to consider the case $z_K > z^*$. Again, from (4) we have

$$(u_K^*(x) - d)\frac{d^s J_K(x)}{dx} = - g(x) + (\lambda_K + \gamma)J_K(x) - \lambda_K J_{K-1}(x) - \mu_K(J_{K+1}(x) - J_K(x)). \quad (10)$$

Since $z_{K+1} \le z_K$, there are two cases to consider:

1) $z_{K+1} = z_K$. Since $z_{K+1} = z_K > z^*$, it follows from ii) that $J_{K+1}(x) - J_K(x)$ is increasing for $x \in [z_K - \Delta_{K+1}, z_K)$. Hence similar to the case $n = N$, we can show, based on (10), that $J_{K-1}(x)$ is decreasing and $J_K(x) - J_{K-1}(x)$ is increasing for $x \in [z_K - \Delta_K, z_K)$, where $\Delta_K = \min\{\Delta_{K+1}, z_K - z^*\} > 0$.
2) $z_{K+1} < z_K$. Since z_K and z_{K+1} are minimum points of $J_K(x)$ and $J_{K+1}(x)$ respectively, which implies that $J_{K+1}(x) - J_K(x)$ is increasing for $x \in [z_{K+1}, z_K)$. Therefore, as in Case 1 we can show that $J_{K-1}(x)$ is decreasing and $J_K(x) - J_{K-1}(x)$ is increasing for $x \in [z_K - \Delta_K, z_K)$, where $\Delta_K = z_K - z_{K+1}$.

This completes the proof. Q.E.D.

The following result follows from Theorem 1.

Corollary 1: If $\overline{u}_k \ge d$ $(k = k_0, \cdots, N)$, then $z_{k_0} \ge \cdots \ge z_N \ge z^*$.

IV. An Extension

In this section, we consider a more complicated system than the one discussed in the previous two sections. For simplicity and illustrative purpose, let us first consider the case $N = 1$. Now the failure time at machine state one is equal to the sum of s_1 independent exponential random variables with rates $\lambda_{11}, \cdots, \lambda_{1s_1}$, or in other word, the machine has s_1 failure stages at state one. Equivalently, we can think that the system has s_1 substates within machine state one. Let $J_{1i}(x)$ be the minimum cost when the system starts at $x(0) = x$ and stage i. By definition, we have $J_{1s_1}(x) = J_0(x)$. We further assume that the system (or the machine) has a maximum production capacity $\overline{u}_{1i}$ at stage i. Then the HJB equation can be written as

$$\min_{0 \le u_{1i} \le \overline{u}_{1i}} (u_{1i} - d)\frac{dJ_{1i}(x)}{dx} = - g(x) + (\lambda_{1i} + \gamma)J_{1i}(x) - \lambda_{1i}J_{1(i+1)}(x) \quad (11)$$

for $i = 1, \cdots, s_1 - 1$, and

$$-d\frac{dJ_0(x)}{dx} = -g(x) + (\mu_0 + \gamma)J_0(x) - \mu_0 J_{11}(x). \quad (12)$$

Using the methodology of Section III, we can show $J_{1i}(x)$ is a convex function. We can, therefore, replace the derivative of $J_{1i}(x)$ in (11) and (12) by its left or right derivative. Also, the optimal control is the hedging point policy. Let z_{1i} be the optimal hedging point at stage i $(i < s_1)$. Then we have the following theorem.

Theorem 2: If $\overline{u}_{1i} \ge d$ for $i = 1, \cdots, s_1 - 1$, then $z_{11} \le z_{12} \le \cdots \le z_{1(s_1-1)}$.

The proof is similar but much simpler than that of Theorem 1. We shall omit it here. If $\lambda_{11} = \lambda_{12} = \cdots = \lambda_{1s_1}$, then the failure time has an Erlang distribution. Note that when the number of stages in the Erlang distribution goes to infinity while its mean remains a constant, the failure time converges (weakly) to a deterministic variable. Therefore, Theorem 2 indicates that the value of the hedging point should increase as the age of a deterministic failure time increases. We submit this observation should hold for a broader class of age depend failure times (e.g., the failure rate is an increasing function of age such as the Erlang distribution). In fact, this type of result would be more interesting since the stages of the machine are often not observable while the age of the machine is. Unfortunately, we were not able to establish it at this point except for the deterministic case (for a more detailed discussion on the deterministic case, see [9]).[2]

Theorem 2 can be easily extended to general N. We summarize the result in the following theorem.

Theorem 3: Consider the system in Section II. We now allow the system to have S_k failure stages at machine state k with failure rate $\lambda_{n1}, \cdots, \lambda_{ns_n}$ respectively. If the maximum production rate at each stage and each machine state is not less than the demand rate d, then the optimal control is the hedging point policy and the value of the optimal hedging points increases as the state number decreases or the stage number increases.

In fact, Theorem 2 can be further generalized as follows.

Theorem 4: Consider the system in Section II with $N + 1$ states $0, 1, \cdots, N$, where state 0 is the complete failure state with zero capacity. Suppose the state process is an irreducible Markov process with transition rate q_{ij} $(i, j = 0, 2!)$. If $q_{ij} = 0$ when $i - j \ge 2$ and the maximum production rate at each state is not less the demand rate, then the optimal control is the hedging point policy and the value of the optimal hedging points increases as the state number decreases.

Clearly, Theorem 3 is a special case of Theorem 4 in which there are $\sum_{n=1}^{N} s_n + 1$ "states" (each failure stage is now treated as a state).

V. Summary and Future Research Suggestions

We have shown the monotonicity of the value of optimal hedging points with respect to the failure stage number at each machine state

[2]The issue of monotonicity with respect to age is being addressed in our very recent work [10].

and the machine state number. Intuitively, our result shows that the value of optimal hedging points should increase as the system gets closer to the complete failure (zero capacity) state. This kind of structural properties of the optimal control can reduce the control space over which we want to search for the optimal control, and greatly facilitate the process of finding near-optimal controls.

Three possible avenues for extending our work are:

1) Establishing a result similar to that of Theorem 2 with respect to the age of the machine (how long the machine has been in operation). As we already mentioned in Section IV, the failure stages of a machine are merely mathematical states and are often not observable. Therefore, it would be more interesting if we could show that the hedging point increases as the age of the machine increases. So far, we have only proven this for the deterministic failure time. We believe this should, however, hold for far more general cases (see footnote 2).

2) Extending the results of this paper to more complicated systems, such as systems with multiple part-types. Conceivably, it will be much harder for the multiple part-types systems since we need to know not only relative positions among hedging points at different states but also relative positions among production switching surfaces.

3) Applying the structure properties derived here to find optimal or near-optimal controls. One obvious way is to limit ourselves to those controls satisfying these properties while searching for an optimal or near-optimal control. It is also possible to establish some equivalent relationships between the systems under controls with these special structures and some stochastic processes with which we are familiar (e.g., some classic queueing systems), e.g., see [8].

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Computational Aspects of the Product-of-Exponentials Formula for Robot Kinematics

Frank C. Park

***Abstract*— In this article we investigate the modeling and computational aspects of the product-of-exponentials (POE) formula for robot kinematics. While its connections with Lie groups and Lie algebras give the POE equations mathematical appeal, little is known regarding its usefulness for control and other applications. We show that the POE formula admits a simple global interpretation of an open kinematic chain and possesses several useful device-independent features absent in the Denavit–Hartenberg (DH) representations. Methods for efficiently computing the forward kinematics and Jacobian using these equations are presented. In particular, the computational requirements for evaluating the Jacobian from the POE formula are compared to those of the recursive methods surveyed in Orin and Schrader [5].**

I. INTRODUCTION

Robotic control systems that depend on sensory information from the environment are often naturally implemented in terms of end-effector coordinates. In such cases the transformation between end-effector and joint coordinates is necessary, since the control inputs are in the form of torques applied to the joints. In [2] Brockett shows that the equations for an open kinematic chain containing either revolute or prismatic joints can be expressed as a product of matrix exponentials. Because of its compact representation and its connection with Lie groups and Lie algebras, the product-of-exponentials (POE) formula has proven to be a useful tool in kinematic theory [6], [8]. While this formula clearly has theoretical appeal, its effectiveness as a modeling and computational tool for

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The author is with the Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92717.

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