The Departure Process of the GI/G/1 Queue and Its MacLaurin Series

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Abstract

In this paper, we study the departure process of the GI/G/1 queue. We develop a simple recursive procedure to calculate the MacLaurin series of its moments and covariances with respect to a parameter in the service time. Based on this recursive procedure the explicit formulas of the coefficients of these MacLaurin series can be obtained in terms of derivatives of the probability density function of the interarrival time evaluated at zero and the moments of the interarrival time and the service time. One important application of these MacLaurin series is that they can be used to obtain the entire response curves of the moments and variances of the departure process, for example, via interpolation by polynomials or rational functions.

KEY WORDS: GI/G/1 queue; departure process; moments; covariances; MacLaurin series

One important issue in the study of queueing systems is to characterize departure processes. Study on departure processes was first initiated by Burke (1956), who shows that the departure process of the M/M/1 queue is a Poisson process. Since then many researchers have worked on the subject; e.g., see Reynolds (1975), Daley (1976), Disney and König (1985), Whitt (1984), and references therein. It is now well recognized that departure processes are very complicated point processes and they are extremely difficult to analyze except for a few special cases (e.g., the M/M/1 queue). Almost all existing analytical results on moments and covariances of departure processes are obtained for simple queues with exponential interarrival and/or service times; see Daley (1968, 1976), Reynolds (1975), Disney and de Morais (1976), Disney and König (1985), Stanford et al. (1987), and references therein. One common approach that is often used is to approximate departure processes, especially in the study of queueing networks, by renewal processes based on their first several (often first two) moments; e.g., see Whitt (1982, 1983).

In this paper, we study the departure process of the GI/G/1 queue. Our objective is to obtain its moments as well as its covariances. Our method is mainly based on a recent paper by Gong and Hu (1992), who use MacLaurin series to calculate the moments of the system time and waiting time of the GI/G/1 queue. To obtain MacLaurin series of the moments and covariances of the departure process, we first establish relationships between the moments and covariances of the departure process and the moments of the system and waiting times. Based on these relationships and the results of Gong and Hu (1992), a simple recursive procedure can then be developed to calculate the coefficients of the MacLaurin series of the moments and covariances of the departure process of the GI/G/1 queue with respect to a (scale) parameter in the service time (i.e., the derivatives when the service time is equal to zero). We point out that some of the relationships used in this paper are known and have also been used by others to study the departure process. For example, Marshall (1968) and Whitt (1984) used the relationship between the moments of the departure process and the moments of the waiting time to obtain approximations and bounds for the moments of the departure process, on the other hand we use it to obtain the MacLaurin series. However, the relationship derived in this paper between the covariances of the departure process and the moments of the system time is new and it is interesting in its own right.

The significance of our results is that this is the first time that analytical expressions (the MacLaurin series) have been obtained for the moments and covariances of the departure process of the GI/G/1 queue explicitly in terms of known quantities (the service and interarrival times). The coefficients of the MacLaurin series we obtain can be used to interpolate
the entire response curves of the moments and covariances, say, by polynomials or rational functions; see Gong and Hu (1992), Gong and Yan (1991), and Gong, Nanamukul, and Yan (1991). They can also be used in conjunction with heavy traffic limits; see Burman and Smith (1983, 1986), Reiman and Simon (1988), and Fendick and Whitt (1989). (In heavy traffic the departure process of the GI/G/1 is simply a renewal process whose interdeparture time is equal to the service time.)

The remainder of this paper is organized as follows. Some preliminary results are given in Section 1. In Section 2, we derive the recursive formulas for the moments and covariances of the departure process. In Section 3, we address some theoretical issues related to our method. Section 4 is a conclusion.

1 Preliminaries

Consider the GI/G/1 queue, in which the arrivals form a renewal process and the service times are i.i.d. (independently and identically distributed) and are also independent of the arrival process. Unless otherwise stated, in the rest of this paper we shall always assume that the GI/G/1 queue under consideration is stable, and we also assume that the queue starts in steady state. We first establish the following notation.

- \( S_n \) the service time of job \( n \),
- \( A_n \) the interarrival time between jobs \( n \) and \( n + 1 \),
- \( T_n \) the system time of job \( n \),
- \( W_n \) the waiting time of job \( n \),
- \( D_n \) the interdeparture time between jobs \( n \) and \( n + 1 \),
- \( S \) a generic service time,
- \( A \) a generic interarrival time,
- \( f(x) \) the p.d.f. (probability density function) of \( A \),
- \( T \) a generic steady-state system time,
- \( W \) a generic steady-state waiting time,
- \( D \) a generic steady-state interdeparture time.
(Note that $T_n$, $W_n$, and $D_n$ are equal in distribution to $T$, $W$, and $D$, respectively.)

Gong and Hu (1992) recently developed a simple recursive procedure to calculate the coefficients of the MacLaurin series of the moments of $T$ and $W$. The basic idea there can be illustrated as follows. First we introduce a scale parameter $\theta$ into the service time, i.e., we consider a GI/G/1 queue with interarrival time $A$ and service time $\theta S$. It is clear that the parameterized queue reduces to the original queue when $\theta = 1$. For convenience, we still use the same notation introduced at the beginning of this section for this new queue; however, we should note that all the quantities except the interarrival times are now functions of $\theta$.

Since

$$T_{n+1} = \theta S_{n+1} + W_{n+1} = \theta S_{n+1} + \max(T_n - A_n, 0),$$  

we have

$$T = \theta S + W$$

$$d = \theta S + \max(T - A, 0),$$

where $d$ means equal in distribution. Note that in (1), $T_n$, $A_n$, and $S_{n+1}$ are independent of each other; therefore, $T$, $A$ and $S$ in (3) are independent of each other as well. It then immediately follows from (2) that

$$\frac{E[T^k]}{k!} = \sum_{j=0}^{k} \beta_{k-j} \frac{E[W^j]}{j!} \theta^{k-j},$$

where $\beta_k \triangleq E[S^k]/k!$.

On the other hand, supposing $f(x)$ can be expanded as

$$f(x) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} x^j, \quad \text{for } x \in [0, \infty),$$

where $\alpha_j \triangleq f^{(j)}(0^+)$ is the $j$th right-hand side derivative of $f(x)$ at $x = 0$, we then have

$$\frac{E[W^k]}{k!} = E \int_0^x \frac{(T - x)^k}{k!} f(x) dx$$

$$= E \int_0^x \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} (T - x)^k x^j dx$$

$$= \sum_{j=0}^{\infty} \frac{\alpha_j}{(k+j+1)!} E[T^{k+j+1}], \quad k = 1, 2, \ldots$$

In the above derivation, we have assumed that expectation, integral, and summation operations can be exchanged freely; technical conditions will be discussed in Section 3. We
write
\[ E[T^k]/k! = \sum_{m=0}^{\infty} t_{km} \theta^m \] (6)
and
\[ E[W^k]/k! = \sum_{m=0}^{\infty} w_{km} \theta^m. \] (7)

We now substitute (6) and (7) into (4) and (5) repeatedly and compare the corresponding coefficients of \( \theta^m, m = 0, 1, 2, \ldots \), which leads to
\[ t_{km} = \begin{cases} \beta_k, & m = k; \\ \sum_{i=1}^{k} \beta_{k-i} w_{k-i} u_{m-k-i}, & m > k; \\ 0, & m < k; \end{cases} \] (8)
\[ w_{km} = \begin{cases} \sum_{i=0}^{m-k-1} \alpha_i t(k_{i+1+1} m), & m > k; \\ 0, & m \leq k \end{cases} \] (9)
for \( k = 1, 2, \ldots \).

2 The Departure Process

In this section we discuss how to obtain the MacLaurin series of the moments and covariances of the departure process of the GI/G/1 queue. We focus on deriving explicit formulas for the coefficients of the MacLaurin series. We postpone our discussion on some technical issues, which are related to our derivation, to the next section.

2.1 The Moments

Since
\[ D_n = \max(A_n - T_n, 0) + \theta S_n + 1, \] (10)
we have
\[ D \overset{d}{=} \max(A - T, 0) + \theta S. \] (11)

Note in (10) that \( A_n, T_n, \) and \( S_{n+1} \) are independent of each other; therefore, \( A, T, \) and \( S \) in (11) are independent of each other as well. Based on (11), we have
\[ E[D^k]/k! = \sum_{j=0}^{k} \beta_{k-j} \theta^{k-j} E[(\max(A - T, 0))^j]/j!. \]

We also have
\[ (\max(A - T, 0))^j = (-1)^j((T - A)^j - (\max(T - A, 0))^j) = (-1)^j((T - A)^j - W^j), \]
for $j = 1, \ldots, k$. Thus,
\[
\begin{align*}
\frac{E[D^k]}{k!} &= \beta_k \theta^k + \sum_{j=1}^{k} \beta_{k-j} \theta^{k-j} \frac{(-1)^j E[(T - A)^j] - W^j}{j!} \\
&= \sum_{j=0}^{k} \beta_j \gamma_{k-j} \theta^j + \sum_{j=1}^{k} (-1)^j \beta_{k-j} \theta^{k-j} \left( \sum_{i=1}^{j} (-1)^{j-i} \gamma_{j-i} \frac{E[T^i]}{t^i} - \frac{E[W^j]}{j!} \right),
\end{align*}
\]

(12)

where $\gamma_k \triangleq E[A^k]/k!$. It should be pointed out that formulas similar to (12) have also been used by Marshall (1968) and Whitt (1982) to derive approximations and bounds of the moments of $D$ based on approximations and bounds of the moments of $W$ and $T$. Here we use (12) to derive MacLaurin series of the moments of $D$. We write
\[
\frac{E[D^k]}{k!} = \sum_{m=0}^{\infty} d_{km} \theta^m.
\]

It immediately follows from (12) that
\[
d_{km} = \begin{cases} 
\sum_{j=\max(1,k-m)}^{k} (-1)^j \beta_{k-j} \left( \sum_{i=1}^{j} (-1)^{j-i} \gamma_{j-i} \frac{E[T^i]}{t^i} - \frac{E[W^j]}{j!} \right) + \beta_m \gamma_{k-m}, & m \leq k; \\
\sum_{j=\max(1,k-m)}^{k} (-1)^j \beta_{k-j} \left( \sum_{i=1}^{j} (-1)^{j-i} \gamma_{j-i} \frac{E[T^i]}{t^i} - \frac{E[W^j]}{j!} \right), & m > k.
\end{cases}
\]

(13)

We now consider the case $k = 2$. Based on (13) (also using (8) and (9)), we have
\[
\begin{align*}
d_{00} &= \frac{E[A^2]}{2} \\
d_{21} &= 0 \\
d_{22} &= E[S^2] \left( 1 - \frac{\alpha_0 E[A]}{2} \right) - (E[S])^2 \\
d_{2m} &= -E[A] w_{1m} + E[S] w_{1(m-1)} & \text{for } m > 2.
\end{align*}
\]

If the arrival process is a Poisson process with rate $\lambda$, then $\alpha_j = \lambda (-\lambda)^j$ and $E[A] = 1/\lambda$. Therefore, based on (8) and (9) we have
\[
\begin{align*}
w_{1m} &= \sum_{i=0}^{m-2} \alpha_i t_{(2+i)m} \\
&= \alpha_0 E[S] w_{1(m-1)} + \alpha_0 w_{2m} + \sum_{i=1}^{m-2} \alpha_i t_{(2+i)m} \\
&= \alpha_0 E[S] w_{1(m-1)} + \sum_{i=0}^{m-3} \left( \alpha_0 \alpha_i + \alpha_i+1 \right) t_{(3+i)m} \\
&= \alpha_0 E[S] w_{1(m-1)},
\end{align*}
\]

which leads to $d_{2m} = 0$ for $m > 2$. Hence, for the M/G/1 queue we obtain
\[
E[D^2] = E[A^2] + (E[S^2] - 2(E[S])^2) \theta^2.
\]

(14)
Furthermore, for the M/M/1 queue we have $E[S^2] = 2(E[S])^2$, which gives $E[D^2] = E[A^2]$. We point out that the second moment of the interarrival time for the M/G/1 queue given by (14) is well-known in the literature, e.g., (3.3a) in Daley (1976). However, our derivation given above is new.

2.2 The Covariances

We now show how to obtain the MacLaurin series of the covariances of the departure process. It is not difficult to observe that

$$D_n = T_{n+1} - T_n + A_n,$$

which gives us

$$\text{Cov}(D_1, D_n) = E[D_1 D_n] - E[D_1] E[D_n]$$
$$= 2E[T_1 T_n] - E[T_1 T_{n-1}] - E[T_1 T_{n+1}] - E[A_1 T_n] + E[A_1 T_{n+1}], \quad (15)$$

for $n \geq 2$. Note $E[D_n] = E[A]$, $E[T_n] = E[T_{n+1}] = E[T_1]$, $E[T_2 T_{n+1}] = E[T_1 T_n]$, $E[T_2 T_n] = E[T_1 T_{n-1}]$, and $E[T_1 A_n] = E[T_2 A_n] = E[T] E[A]$. Therefore, we only need to obtain MacLaurin series of $E[T_1 T_{n-1}]$, $E[T_1 T_n]$, $E[T_1 T_{n+1}]$, $E[A_1 T_n]$, and $E[A_1 T_{n+1}]$. Our basic idea is to express them in terms of $E[T^k]$, $k = 1, 2, \ldots$.

In the remainder of this section, we shall focus mainly on the case $n = 2$ since $\text{Cov}(D_1, D_2)$ is the most interesting covariance and also since the basic idea used can be easily applied to $\text{Cov}(D_1, D_n)$ ($n > 2$) though the calculations become messier. For $n = 2$, (15) becomes

$$\text{Cov}(D_1, D_2) = 2E[T_1 T_2] - E[T^2] - E[T_1 T_2] - E[A_1 T_2] + E[A_1 T_3]. \quad (16)$$

Since $W_{n+1} = \max(T_n - A_n, 0)$, similar to (5) we have

$$E[T_1 W_{2}^2] = k! \sum_{j=0}^{\infty} \alpha_j \frac{E[T^k+j+2]}{(k+j+1)!}; \quad (17)$$

and

$$E[A_1 W_{2}^2] = k! \sum_{j=0}^{\infty} \alpha_j \frac{(j+1) E[T^k+j+2]}{(k+j+2)!}. \quad (18)$$

Based on (17) and (18), we obtain

$$\frac{E[T_1 T_2^k]}{k!} = \frac{E[T_1 W_2 + \theta S_2^k]}{k!} = E[T] \beta_k \theta^k + \sum_{i=1}^{k} \beta_{k-i} \theta^{k-i} \sum_{j=0}^{\infty} \alpha_j \frac{E[T^{i+j+2}]}{(i+j+1)!}, \quad (19)$$

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\[
\frac{E[A_1 T_{2}^k]}{k!} = \frac{E[A_1 (W_2 + \theta S_2)^k]}{k!} = E[A] \beta_k \theta^k + \sum_{i=1}^{k} \beta_k \theta^{k-i} \sum_{j=0}^{\infty} \alpha_j (j + 1) E[T_{i+j+2}^j] (i + j + 2)!.
\] (20)

Denote
\[
\frac{E[T_1 T_{2}^k]}{k!} = \sum_{m=0}^{\infty} t_{km}^{(12)} \theta^m \quad \text{and} \quad \frac{E[A_1 T_{2}^k]}{k!} = \sum_{m=0}^{\infty} a_{km}^{(12)} \theta^m.
\]

Then it immediately follows from (19) and (20) that
\[
t_{km}^{(12)} = \begin{cases} 
\beta_k t_1 (m-k) + \sum_{i=1}^{k} \beta_k t_{i} (m-k-i) \sum_{j=0}^{\infty} \alpha_j (i+j+2) t_{(i+j+2)(m-k+i)}, & m > k + 1; \\
\beta_k t_1 (m-k), & m = k + 1; \\
0, & m \leq k;
\end{cases}
\] (21)

\[
a_{km}^{(12)} = \begin{cases} 
\sum_{i=1}^{k} \beta_k t_{i} (m-k) \sum_{j=0}^{\infty} \alpha_j (i+j+2) t_{(i+j+2)(m-k+i)}, & m > k + 1; \\
0, & m = k + 1; \\
E[A] \beta_k, & m = k; \\
0, & m < k.
\end{cases}
\] (22)

Similar to (17)-(20), we have for \( n \geq 2 \)
\[
E[T_1 W_{n+1}^k] = k! \sum_{j=0}^{\infty} \alpha_j E[T_1 T_{n+1}^{i+j+1}] (k + j + 1)!;
\]

\[
E[A_1 W_{n+1}^k] = k! \sum_{j=0}^{\infty} \alpha_j E[A_1 T_{n+1}^{i+j+1}] (k + j + 1)!
\]

\[
E[T_1 T_{n+1}^k] = E[T_1] \beta_k \theta^k + \sum_{i=1}^{k} \beta_k \theta^{k-i} \sum_{j=0}^{\infty} \alpha_j E[T_1 T_{n+1}^{i+j+1}] (i + j + 1)!
\]

\[
E[A_1 T_{n+1}^k] = E[A] \beta_k \theta^k + \sum_{i=1}^{k} \beta_k \theta^{k-i} \sum_{j=0}^{\infty} \alpha_j E[A_1 T_{n+1}^{i+j+1}] (i + j + 1)!
\]

Denote
\[
\frac{E[T_1 T_{n}^k]}{k!} = \sum_{m=0}^{\infty} t_{km}^{(1n)} \theta^m \quad \text{and} \quad \frac{E[A_1 T_{n}^k]}{k!} = \sum_{m=0}^{\infty} a_{km}^{(1n)} \theta^m.
\]

We have
\[
t_{km}^{(1(n+1))} = \begin{cases} 
\beta_k t_1 (m-k) + \sum_{i=1}^{k} \beta_k t_{i} (m-k-i) \sum_{j=0}^{\infty} \alpha_j t_{(i+j+1)(m-k+i)}, & m > k + 1; \\
\beta_k t_1 (m-k), & m = k + 1; \\
0, & m \leq k;
\end{cases}
\] (23)

\[
a_{km}^{(1(n+1))} = \begin{cases} 
\sum_{i=1}^{k} \beta_k t_{i} (m-k) \sum_{j=0}^{\infty} \alpha_j a_{(i+j+1)(m-k+i)}, & m > k + 1; \\
0, & m = k + 1; \\
E[A] \beta_k, & m = k; \\
0, & m < k.
\end{cases}
\] (24)
Finally, based on (16) we have
\[
d_{m}^{12} = \begin{cases} 
2t_{1m}^{(12)} - 2t_{2m} - t_{1m}^{(13)} - a_{1m}^{(12)} + a_{1m}^{(13)}, & m \geq 1; \\
0, & m = 0,
\end{cases} \tag{25}
\]
where \(\sum_{m=0}^{\infty} d_{m}^{(12)} \theta^{m} \) is the MacLaurin series of \( \text{Cov}(D_{1}, D_{2}) \).

With (25) we calculate the first four coefficients
\[
d_{0} = (E[A])^{2} \\
d_{1} = 0 \\
d_{2} = (E[S])^{2} - \left(1 - \frac{\alpha_{0}E[A]}{2}\right) E[S^{2}] \\
d_{3} = \left(\frac{\alpha_{0}}{2} + \frac{\alpha_{1}E[A]}{6}\right) E[S^{3}] - \alpha_{0}E[S]E[S^{2}].
\]
It is clear that for the M/M/1 queue we have \(d_{2} = d_{3} = 0\).

Similar to (25), for \(n > 2\) we have
\[
d_{m}^{1n} = \begin{cases} 
2t_{1m}^{(1n)} - t_{2m}^{(1(n-1))} - t_{1m}^{(1(n+1))} - a_{1m}^{(1n)} + a_{1m}^{(1(n+1))}, & m \geq 1; \\
0, & m = 0,
\end{cases} \tag{26}
\]
where \(\sum_{m=0}^{\infty} d_{m}^{(1n)} \theta^{m} \) is the MacLaurin series of \( \text{Cov}(D_{1}, D_{n}) \) and \(t_{1m}^{(1n)}, t_{2m}^{(1(n-1))}, t_{1m}^{(1(n+1))}, a_{1m}^{(1n)}\) and \(a_{1m}^{(1(n+1))}\) can be calculated recursively via (21)-(24).

It is worth pointing out that the above procedure can also be used to obtain MacLaurin series of covariance \( \text{Cov}(D_{1}^{k_{1}}, D_{2}^{k_{2}}) \) (where \(k_{1}\) and \(k_{2}\) are two non-negative integers).

3 Some Theoretical Issues

In this section, we address some theoretical issues regarding the coefficients of the MacLaurin series we obtained previously. Mainly, we need to know 1) whether the moments and covariances of the departure process have all the derivatives with respect to \(\theta\) at \(\theta = 0^{+}\), and 2) whether these derivatives can be calculated based on the formulas we derived in the previous section.

We impose the following two conditions.

(A1). All the moments of \(S\) are finite, and \(E[S^{k}] \leq k!(C_{s})^{k}\) for \(k = 0, 1, 2, \ldots\), where \(C_{s} > 0\) is a constant;

(A2). \(f(x)\) is an analytic at \(x = 0^{+}\), i.e., there exists \(x_0 > 0\) such that
\[
f(x) = \sum_{j=0}^{\infty} \frac{\alpha_{j}}{j!} x^{j} \text{ for } 0 \leq x \leq x_0.
\]
(Recall $\alpha_j$ is the $j$-th right-hand side derivative of $f(x)$ at $x = 0$.) Furthermore, $|\alpha_j| \leq (C_f)^{j+1}$ ($j = 1, 2, \ldots$), and

$$\left| \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} x^j \right| \leq C_f,$$

and $f(x) \leq C_f$, for $x \geq 0$,

where $C_f > 0$ is a constant.

We note that (A1) and (A2) are very general and they are satisfied by almost all probability distributions used in queueing theory, such as phase-type, uniform, and deterministic.

To establish differentiability of the moments and covariances, we need the following two lemmas.

**Lemma 1.** (A1) holds if and only if $S$ has a finite geometric moment, i.e., there is a $\rho > 1$ such that $E[\rho^S] < \infty$.

**Proof.** If (A1) holds, choosing $\theta < 1/C_s$, we then have

$$\sum_{k=0}^{\infty} \frac{E[S^k]}{k!} \theta^k \leq \sum_{k=0}^{\infty} (C_s)^k \theta^k = \frac{1}{1 - C_s \theta} < \infty.$$

Let $\rho = e^\theta > 1$. By applying the dominated convergence theorem, we have

$$E[\rho^S] = \sum_{k=0}^{\infty} \frac{E[S^k]}{k!} \theta^k < \infty.$$

On the other hand, if $E[\rho^S] < \infty$, we choose $\theta = \ln \rho > 0$. It is clear that the sequence $\{\sum_{k=0}^{K} E[S^k] \theta^k / k!, K = 0, 1, 2, \ldots\}$ is monotonically increasing and bounded by $E[e^{\theta S}]$, hence its limit exists, i.e., $\sum_{k=0}^{\infty} E[S^k] \theta^k / k! < \infty$, therefore (A1) must hold. Q.E.D.

**Lemma 2.** If (A1) holds, then $E[T(\theta)^k] \leq k!(\theta C_T)^k$, where $C_T$ is a constant and $0 \leq \theta \leq 1$, and

$$\lim_{\theta \to 0} \frac{1}{\theta^m} E[T(\theta)^k 1(T(\theta) > x_0)] = 0,$$

for any non-negative integers $k$ and $m$, where $1(\cdot)$ denotes the indicator function.

**Proof.** (A1) implies $T(\theta)$ has a finite geometric moment (see Thorisson 1985), hence $E[T(1)^k] \leq k!(C_T)^k$ (Lemma 1). On the other hand, according to Weber (1983), $T(\theta)$ is a.s. (almost surely) a convex function of $\theta$. Also note $T(0) = 0$, therefore, $T(\theta) \leq \theta T(1)$ a.s. for $0 \leq \theta \leq 1$. Hence $E[T(\theta)^k] \leq k!(\theta C_T)^k$.

We now prove the second part of the lemma. Based on the first part, it is clear that we only need to consider the case $k \leq m$. Since $T(\theta) \leq \theta T(1)$ a.s. for $0 \leq \theta \leq 1$, we have

$$\frac{1}{\theta^m} T(\theta)^k 1(T(\theta) > x_0) \leq x_0^{k-m} T(1)^m 1(T(1) > x_0 / \theta).$$
Note that $E[T(1)^m] < \infty$. Therefore,
\[
\lim_{\theta \to 0} \frac{1}{\theta^n} E[T(\theta)^k 1(T(\theta) > x_0)] \leq \lim_{\theta \to 0} x_0^{k-m} E[T(1)^m 1(T(1) > x_0/\theta)] = 0.
\]

Q.E.D.

With Lemmas 1 and 2, we are now ready to show that the moments and covariances of the departure process have all the derivatives at $\theta = 0^+$.

**Theorem 1.** Under (A1) and (A2), $E[T^k]$ and $E[W^k]$ ($k = 1, 2, \ldots$) have derivatives of any order at $\theta = 0^+$, and
\[
\begin{align*}
\frac{d^m E[T^k]}{d\theta^m} & \bigg|_{\theta = 0^+} = k! m! t_{km}, \\
\frac{d^m E[W^k]}{d\theta^m} & \bigg|_{\theta = 0^+} = k! m! w_{km},
\end{align*}
\]
where $t_{km}$ and $w_{km}$ are defined by (8) and (9).

**Proof.** According to (A2), we first have
\[
\frac{E[W^k]}{k!} = E \left[ \int_0^T \frac{(T-x)^k}{k!} \sum_{j=0}^{\infty} \frac{\alpha_j x^j}{j!} \, dx \right] + E \left[ 1(T > x_0) \int_0^T \frac{(T-x)^k}{k!} \left( f(x) - \sum_{j=0}^{\infty} \frac{\alpha_j x^j}{j!} \right) \, dx \right].
\]
Since
\[
\sum_{j=m-k}^{\infty} |\alpha_j| \frac{E[T^{k+j+1}]}{(k+j+1)!} \leq \sum_{j=m-k}^{\infty} (C_f)^j (\theta C_T)^{k+j+1} = \frac{(C_f)^{m-k} (\theta C_T)^{m+1}}{1 - \theta C_f C_T},
\]
for $\theta < 1/(C_f C_T)$, by applying the dominated convergence theorem twice (one for interchanging the summation and the integration and the other for interchanging the summation and the expectation), we have
\[
\left| E \int_0^T \frac{(T-x)^k}{k!} \sum_{j=m-k}^{\infty} \frac{\alpha_j x^j}{j!} \, dx \right| \leq \frac{(C_f)^{m-k} (\theta C_T)^{m+1}}{1 - \theta C_f C_T}.
\]

Also note
\[
\lim_{\theta \to 0} \frac{1}{\theta^n} E \left[ 1(T > x_0) \int_0^T \frac{(T-x)^k}{k!} \left( f(x) - \sum_{j=0}^{\infty} \frac{\alpha_j x^j}{j!} \right) \, dx \right] \leq \lim_{\theta \to 0} \frac{1}{\theta^n} 2C_f E[1(T > x_0)T^{k+1}] = 0.
\]
Therefore,

\[
\frac{E[W^k]}{k!} = \sum_{j=0}^{m-k-1} \frac{E[T^k+j+1]}{(k+j+1)!} + o(\theta^m).
\]

Combining this with

\[
\frac{E[T^k]}{k!} = \sum_{j=0}^{k} \beta_{k-j} \frac{E[W^j]}{j!} \theta^{k-j},
\]

we can easily prove the theorem, say, via induction. Q.E.D.

**Corollary 1.** Under (A1) and (A2), all the moments and covariances have derivatives of any order at \( \theta = 0^+ \). In particular, we have

\[
\frac{d^m E[D^k]}{d \theta^m} \bigg|_{\theta = 0^+} = k! m H_{km},
\]

\[
\frac{d^m \text{Cov}(D_1, D_n)}{d \theta^m} \bigg|_{\theta = 0^+} = m d_m^{(1n)},
\]

where \( d_{km} \) and \( d_m^{(1n)} \) are defined by (13) and (26) ((25) for \( n = 2 \)) respectively.

*Proof.* Differentiability of the moments falls out directly from (12). For the covariances, the proof will be very similar to that of Theorem 1, hence we omit it here. Q.E.D.

It should be pointed out that the existence of all the derivatives at \( \theta = 0^+ \) does not guarantee the analyticity at \( \theta = 0^+ \), which is a necessary condition for using polynomials or rational functions to obtain the whole response curves based solely on the derivatives at \( \theta = 0^+ \). In fact, Hu (1992) gives two examples in which \( E[W^k] \) and \( E[T^k] \) have all the derivatives at \( \theta = 0^+ \) but are not analytic. Furthermore, it is proven in Hu (1992) that if (A2) is replaced by

\[(A2') \text{ If } f(x) \text{ can be expressed as } \]

\[
f(x) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} x^j \text{ for } x \geq 0,
\]

and \( |\alpha_j| \leq (C_j)^{j+1} (j = 1, 2, \ldots) \),

then \( E[W^k] \) and \( E[T^k] \) are analytic at \( \theta = 0^+ \). This implies that the moments and covariances of the departure process are also analytic at \( \theta = 0^+ \). The proof on the analyticity is more involved and we will not elaborate on it here. The interested reader is referred to Hu (1992).
4 Conclusion

We have derived a simple procedure to calculate the coefficients of the MacLaurin series of the moments and covariances of the departure process of the GI/G/1 queue with respect to \( \theta \), a scale parameter in the service time. Under some mild conditions we show that the moments and covariances have derivatives of any order at \( \theta = 0 \) and can be calculated based on the coefficients we obtained. These derivatives can then be used to interpolate the moments and covariances of the departure process, say, based on polynomials or rational functions. They can also be used in conjunction with heavy traffic limits, which are extremely easy to obtain for the departure process. It is clear that various approximations for the moments and covariances can be developed depending on how many terms of derivatives we use.

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References


