Analyticity of Single-Server Queues in Light Traffic

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Abstract

Recently, several methods have been proposed to approximate performance measures of queueing systems based on their light traffic derivatives, e.g., the MacLaurin expansion, the Padé approximation, and interpolation with heavy traffic limits. The key condition required in all these approximations is that the performance measures be analytic when the arrival rates equal to zero. In this paper, we study the GI/G/1 queue. We show that if the c.d.f. of the interarrival time can be expressed as a MacLaurin series over $[0, \infty)$, then the mean steady-state system time of a job is indeed analytic when the arrival rate to the queue equals to zero. This condition is satisfied by phase-type distributions but not c.d.f.'s without support $[0, \infty)$, such as uniform and shifted exponential distributions. In fact, we show through two examples that the analyticity does not hold for most commonly used distribution functions which do not satisfy this condition.

Keywords: The GI/G/1 queue, light traffic derivatives, MacLaurin series, analyticity
Short Title: Analyticity of the GI/GI/1 queue

1 Introduction

In this paper, we study the analyticity of the mean (in fact, all the moments of) steady-state system time of a job in the GI/G/1 queue in light traffic, i.e., when the arrival rate to the queue, denoted by λ, approaches zero. Our work is mainly motivated by several approximation methods recently developed which use the light traffic derivatives of the mean system time (i.e., the derivatives with respect to λ at $\lambda = 0$) to approximate the entire response curve of the mean system time with respect to λ. In what follows, we first briefly review previous works on the subject.

Benes [2] was among the first to study light traffic derivatives for queueing systems. Since then, many papers have been written on the subject of light traffic derivatives, e.g., see Reiman and Siman [13], Daley and Rolski [5], Asmussen [1], Gong and Hu [7], Sigman [14], and refereces therein. In particular, Gong and Hu's work is closely related to our work in this paper. In Gong and Hu [7], a simple recursive procedure is derived to calculate the light traffic derivatives for the GI/G/1 queue. We shall see that our proof of analyticity is mainly based on some recursive formulas very similar to those derived there.

An immediate application of the light traffic derivatives is to use them to approximate the mean system time via a MacLaurin series. However, the MacLaurin series may result in poor approximations due to its slow convergence rate, especially for heavy traffic cases. Furthermore, as noted in Gong and Hu [7], for many GI/G/1 queues, including some well-behaved queues such as the E_2/M/1 queue with exponential service times and two-stage Erlang interarrival times, the MacLaurin series only converges in a "small" neighborhood of $\lambda = 0$. One way to overcome this problem to use the light traffic derivatives in conjunction with heavy traffic limits, e.g., see Burman and Smith [3, 4], Reiman and Simon [12], and Fendick and Whitt [6]. (In fact, these authors approximate the mean system time by ratios of polynomial functions and a normalization factor $1 - \rho$, where ρ is the traffic intensity.) Another method recently proposed by Gong, Nananukul, and Yan [8] is to use the Padé approximation method to approximate the mean system time based on the light traffic derivatives. Basically, in the Padé approximation, we use *general* rational functions instead of polynomial functions to approximate the response curve of the mean system time. This method seems very promising and the experimental results in Gong, Nananukul, and Yan [8] show that in many cases only a few light traffic derivatives can often yield very good approximations.

The key condition needed in all the approximate methods mentioned above be that the mean system time is analytic at $\lambda = 0$. So far, little is known about the analyticity of queueing systems. To this author's best knowledge, the only result on the analyticity is given by Zazanis [19], in which he proves performance measures defined on stochastic processes driven by Poisson inputs are analytic. His proof is based on a notion called absolute monotonicity. For the GI/G/1 queue considered in this paper, his result in fact becomes a special case of ours. Usually, it is very difficult to show analyticity. We need to prove that (1) derivatives of arbitrary order exist and that (2) the corresponding MacLaurin/Taylor series converges to the correct value.

The way we prove the analyticity can be briefly described as follows. We first construct a sequence of MacLaurin series based on a recursive procedure similar to the one derived in Gong and Hu [7]. (In fact, the one given there is a special case of ours.) We then prove that these MacLaurin series converge in a neighborhood of $\lambda = 0$. Lastly, we show that the moments of the system time are in fact equal to these MacLaurin series. So, we in fact prove that all the moments of the system time are analytic at $\lambda = 0$. The key condition we use in establishing the analyticity is that the c.d.f. (cumulative distribution function) of the interarrival time can be expanded as a MacLaurin series over $[0, \infty)$. This condition is satisfied by all phase-type distributions, but it excludes all c.d.f.'s whose support is not equal to $[0, \infty)$, such as uniform and shifted exponential distribution functions. In fact, through two simple examples we illustrate that the mean system time is usually *not* analytic at $\lambda = 0$ if the c.d.f. of the interarrival time does not have support $[0, \infty)$. Therefore, the condition is perhaps also a necessary one though we are not able to prove it at this point.

The rest of this paper is organized as follows. In Section 2, we present our main result. In Section 3, we discuss two examples, a uniform interarrival time and a shifted exponential interarrival time. We show that the analyticity at $\lambda = 0$ does not hold for these two queues. The proof of our main result is given in Section 4.

2 The Main Result

Consider a first-come-first-served GI/G/1 queue with a renewal arrival process and i.i.d. (independently and identically distributed) service times. Let S be a generic service time and A be a generic interarrival time. Unless otherwise stated, throughout this paper we shall always assume $E[S] < E[A]$ so that the queue is stable and ergodic. Denote the steady-state system time and waiting time of a job by T and W, respectively. (The system time is the time from when a job enters the system to when it leaves the system, and the waiting time is the time from when a job enters the system to when it enters service.) Let λ be the arrival rate. We assume $A = Y/\lambda$, where Y is a random variable independent of λ with $E[Y] = 1$. Then it is clear that T and W are functions of λ, as are their moments $E[T^n]$ and $E[W^n]$ $(n = 1, 2, \ldots)$.

For ease of exposition, we will focus on T/λ and W/λ. Define $\tilde{T} = T/\lambda$ and $\tilde{W} = W/\lambda$. Then $\tilde{T}$ and $\tilde{W}$ are the steady-state system time and waiting time of a job in the GI/G/1 queue with interarrival time Y and service time λS. Suppose Y has c.d.f. $F(\cdot)$. First, we present our main result in the following theorem.

Theorem 1. Suppose

(A1). All the moments of S are finite, and $E[S^n] \leq n!(C_s)^n$ for $n = 0, 1, 2, \ldots$, where $C_s > 0$

is a constant;

(A2). $F(x)$ is an analytic function over $[0,\infty)$, and

$$F(x)=\sum_{n=0}^{\infty}\frac{F^{(n)}(0^+)}{n!}x^n \quad \text{for } x\geq 0,$$

where $F^{(n)}(0^+)$ is the n-th right-hand side derivative of $F(x)$ at $x=0$ satisfying $F(0^+)<1$ and $|F^{(n)}(0^+)|\leq (C_F)^n$ $(n=1,2,\ldots)$, where $C_F>0$ is a constant,

then, $E[\tilde{T}^n]$ and $[\tilde{W}^n]$ $(n=1,2,\ldots)$ are analytic at $\lambda=0$, or more precisely, there exists $\lambda_0>0$ such that for $0\leq\lambda\leq\lambda_0$

$$\frac{E[\tilde{T}^n]}{n!} = \sum_{m=n}^{\infty} a_{nm}\lambda^m, \tag{1}$$

$$\frac{E[\tilde{W}^n]}{n!} = \sum_{m=n}^{\infty} b_{nm}\lambda^m. \tag{2}$$

Furthermore, coefficients $d_{nm}\triangleq a_{nm}-b_{nm}$ and b_{nm} can be calculated based on the following recursive equations,

$$d_{nm} = \begin{cases} \beta_n+\sum_{i=1}^{n-1}\beta_{n-i}b_{ii}, & m=n; \\ \sum_{i=1}^{n-1}\beta_{n-i}b_{i(m-n+i)}, & m>n; \end{cases} \tag{3}$$

$$b_{nm} = \frac{1}{1-\alpha_0}\sum_{i=0}^{m-n-1}\alpha_{i+1}(d_{(n+1+i)m}+b_{(n+1+i)m})+\frac{\alpha_0}{1-\alpha_0}d_{nm},\ m\geq n, \tag{4}$$

for $n=1,2,\ldots$, where $\alpha_i\triangleq F^{(i)}(0^+)$ and $\beta_i\triangleq E[S^i]/i!$ $(i=0,1,2,\ldots)$

The proof of Theorem 1 will be given in Section 4. We should point out that a special case in which $\alpha_0=0$ is considered in Gong and Hu [7], where two recursive equations similar to (3) and (4) are derived for a_{nm} and b_{nm} (instead of d_{nm} and b_{nm}); however, the analyticity is not discussed there. The way in which coefficients d_{nm} and b_{nm} are calculated via recursions (3) and (4) can be best illustrated by Table 1. We note in Table 1 the numbers in the parentheses indicate the sequence in which the corresponding calculations are performed.

Since $E[T^n] = E[\tilde{T}^n]/\lambda^n$ and $E[W^n] = E[\tilde{W}^n]/\lambda^n$, it then immediately follows from Theorem 1 that

Corollary 1. Under the hypotheses of Theorem 1, $E[T^n]$ and $E[W^n]$ are analytic at $\lambda = 0$.

We note most service time distributions used in queueing theory, such as phase-type, uniform, and deterministic, satisfy Condition (A1). Also the following proposition tells us that (A1) is equivalent to the condition that S has finite moment generating function, i.e., there is a $\theta > 0$ such that $E[e^{\theta S}] < \infty$.

Proposition 1. (A1) holds if and only if the moment generating function $E[e^{\theta S}] < \infty$ for some $\theta > 0$.

Proof. If (A1) holds, choosing $\theta < 1/C_s$, we then have

$$\sum_{n=0}^{K} \frac{E[S^n]}{n!}\theta^n \leq \sum_{n=0}^{\infty} (C_s)^n \theta^n = \frac{1}{1 - C_s\theta} < \infty,$$

for any $K = 0, 1, 2, \ldots$. Since

$$e^{\theta S} = \sum_{n=0}^{\infty} \frac{S^n}{n!}\theta^n,$$

by applying the dominated convergence theorem, we have $E[e^{\theta S}] < \infty$.

On the other hand, if $E[e^{\theta S}] < \infty$, then, $E[S^n] < n! E[e^{\theta S}]/\theta^n < \infty$. It is clear that the sequence $\{\sum_{n=0}^{K} E[S^n]\theta^n/n!, K = 0, 1, 2, \ldots\}$ is monotonically increasing and bounded by $E[e^{\theta S}]$, hence its limit exists, and furthermore, by applying the dominated convergence theorem we have $\sum_{n=0}^{\infty} E[S^n]\theta^n/n! < \infty$, hence (A1) must hold. This completes the proof. □

We now examine Condition (A2). Clearly, (A2) is satisfied by all phase-type distributions (see Neuts [11]). On the other hand, it excludes all c.d.f.'s whose support is not $[0, \infty)$, such as uniform, triangle, and shifted exponential distributions. Two examples in the next section show that if Y has either a uniform distribution or a shifted exponential distribution, then $E[T]$ is *not* analytic at $\lambda = 0$. Thus, it appears that (A2) might also be a necessary condition for $E[T]$ to be analytic at $\lambda = 0$, although we are unable to provide a proof at this point.

Before closing this section, we should point out that even though $E[T^n]$ and $E[W^n]$ may not be analytic at $\lambda = 0$ if (A2) is not satisfied, they can still have derivatives of any order at $\lambda = 0$ under very mild conditions. In fact, it is shown in Hu [9] that if $F(x)$ is analytic at $x = 0^+$ (instead of $[0, \infty)$ required in (A2)), then $E[T^n]$ and $E[W^n]$ have derivatives of any order at $\lambda = 0$; also see the two examples given in the next section. Furthermore, these derivatives can be easily calculated through recursions (3) and (4).

3 Two Negative Examples

In this section, we provide two examples in which $E[T^n]$ and $E[W^n]$ are not analytic function at $\lambda = 0$; one is the U/M/1 queue and the other is the (M+d)/M/1 queue (where M+d represents the shifted exponential distribution, i.e., an exponential random variable plus a constant d). For illustrative purpose, we shall focus on $E[T]$. We note for the GI/M/1 queue (see Takacs [15]),

$$E[T] = \frac{1}{\mu(1-\sigma)}, \tag{5}$$

where σ is the unique solution to the following equation in the range $0 \le \sigma < 1$

$$\sigma = E[e^{-\mu(1-\sigma)Y/\lambda}], \tag{6}$$

and μ is the service rate. In fact, T is exponentially distributed with rate $\mu(1-\sigma)$, thus $E[T^n] = n!(E[T])^n$. Note that when $\lambda = 0$, we have $E[T] = 1/\mu$ and $\sigma = 0$, it then follows from (5) that $E[T]$ is analytic at $\lambda = 0$ if and only if σ is analytic at $\lambda = 0$. Therefore, we only need to investigate analyticity of σ at $\lambda = 0$.

3.1 The U/M/1 Queue

Suppose Y is uniformly distributed over $[0, 2]$, i.e., $f(x) = 1/2$ for $0 \le x \le 2$ and 0 otherwise, and S is exponentially distributed with $\mu = 1$. Then, (6) becomes

$$\frac{\sigma}{\lambda} = \frac{1}{2(1-\sigma)} \left(1 - e^{-2(1-\sigma)/\lambda}\right). \tag{7}$$

Since

$$\lim_{\lambda \to 0^+} \frac{e^{-2(1-\sigma)/\lambda}}{\lambda^n} = 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

it is then not too difficult to verify based on (7) that the nth derivative of σ with respect to λ exists at $\lambda = 0$ ($n = 1, 2, \ldots$). Next we want to calculate $d^n\sigma/d\lambda^n|_{\lambda=0}$. Denote

$$\sigma_n \triangleq \frac{1}{n!} \left.\frac{d^n\sigma}{d\lambda^n}\right|_{\lambda=0} \quad \text{for } n = 1, 2, \ldots.$$

Rewrite (7) as

$$2\sigma(1-\sigma)-\lambda=-\lambda e^{-2(1-\sigma)/\lambda}. \tag{8}$$

We note

$$\left.\frac{d^n(\lambda e^{-2(1-\sigma)/\lambda})}{d\lambda^n}\right|_{\lambda=0}=0, \quad \text{for } n=1,2,\ldots.$$

Differentiating (8) and letting $\lambda=0$, we have

$$\sigma_n=\sum_{k=1}^{n-1}\sigma_k\sigma_{n-k}, \quad \text{for } n=2,3,\ldots, \tag{9}$$

and $\sigma_1=1/2$.

Proposition 2.

$$\sigma_n=\frac{(2n-2)!}{2^n n!(n-1)!}, \quad \text{for } n=1,2,\ldots. \tag{10}$$

Thus,

$$\left.\frac{d^n\sigma}{d\lambda^n}\right|_{\lambda=0}=\frac{(2n-2)!}{2^n(n-1)!}.$$

Proof. Let us consider

$$\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\lambda}{2}}=\sum_{n=1}^{\infty}\frac{(2n-2)!}{2^n n!(n-1)!}\lambda^n,$$

where $\lambda\in[0,1/2)$. (Note: the power series converges when $0\le\lambda<1/2$.) We have

$$\begin{aligned}
-\frac{\lambda}{2}+\sum_{n=1}^{\infty}\frac{(2n-2)!}{2^n n!(n-1)!}\lambda^n &= \frac{1}{2}-\frac{\lambda}{2}-\sqrt{\frac{1}{4}-\frac{\lambda}{2}} \\
&= \left(\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\lambda}{2}}\right)^2 \\
&= \left(\sum_{n=1}^{\infty}\frac{(2n-2)!}{2^n n!(n-1)!}\lambda^n\right)^2 \\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{(2k-2)!}{2^k k!(k-1)!}\frac{(2(n-k)-2)!}{2^{n-k}(n-k)!(n-k-1)!}\lambda^n.
\end{aligned}$$

Hence,

$$\frac{(2n-2)!}{2^n n!(n-1)!}=\sum_{k=1}^{n-1}\frac{(2k-2)!}{2^k k!(k-1)!}\frac{(2(n-k)-2)!}{2^{n-k}(n-k)!(n-k-1)!} \quad \text{for } n\ge 2. \tag{11}$$

Clearly, (10) holds for $n=1$. By applying induction (10) then follows immediately from (9) and (11). □

Based on the proof of Proposition 2, we also have

$$\sum_{n=1}^{\infty} \sigma_n \lambda^n = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\lambda}{2}},$$

which is obviously not equal to σ defined by (7). Therefore, σ is not analytic at $\lambda = 0$.

Since $E[T] = 1/(1-\sigma)$, $E[T]$ also has the n-th derivative at $\lambda = 0$, and

$$\left.\frac{d^n E[T]}{d\lambda^n}\right|_{\lambda=0} = \frac{(2n)!}{2^n n!(n+1)!}, \tag{12}$$

for $n = 1, 2, \ldots$. We note that (3) and (4) result in the same formula. In fact, we can also use (12) to show that $E[T]$ is not analytic at $\lambda = 0$.

3.2 The (M+d)/M/1 Queue

We now consider the (M+d)/M/1 queue in which S is exponentially distributed with $\mu = 1$ and $f(x) = ke^{-k(x-d)}$ for $x \geq d$ and 0 otherwise, where $0 \leq d < 1$ and $k = 1/(1-d)$. We then have from (6)

$$\sigma = \frac{\lambda k}{\lambda k + 1 - \sigma} e^{-(1-\sigma)d/\lambda}. \tag{13}$$

Similar to the first example, we can show based on (13) that σ has derivatives of any order at $\lambda = 0$, and

$$\left.\frac{d^n \sigma}{d\lambda^n}\right|_{\lambda=0} = 0, \quad n = 1, 2, \ldots.$$

Hence,

$$\sum_{n=1}^{\infty} \left.\frac{d^n \sigma}{d\lambda^n}\right|_{\lambda=0} \lambda^n = 0,$$

which implies that σ is not analytical at $\lambda = 0$.

4 Proof of Theorem 1

In this section, we prove Theorem 1. Since we consider exclusively the GI/G/1 queue with interarrival time Y and service time λS in this section, for simplicity of notation, we will use T and W (without tilde) to denote its system time and waiting time. Also, we sometimes may write T and W as $T(\lambda)$ and $W(\lambda)$ to emphasize their dependence on λ.

Without loss of generality, we assume C_F is small enough such that $C_F < 1$ and

$$\frac{C_F}{1 - C_F} + \alpha_0 < 1. \tag{14}$$

(Recall $\alpha_0 < 1$.) Otherwise, we consider a GI/G/1 queue with interarrival time CY and service time $C\lambda S$, where $C > C_F$ is a constant such that

$$\frac{C_F}{C - C_F} + \alpha_0 < 1.$$

The system time and waiting time of this GI/G/1 queue are CT and CW, respectively. The c.d.f. of CY is $\tilde{F}(x) = F(x/C)$. It is clear that $\tilde{F}(x)$ is also analytic over $[0, \infty)$; furthermore, $\tilde{F}(0) = F(0) < 1$ and $|\tilde{F}^{(n)}(0)| = |F^{(n)}(0)/C^n| \leq (C_F/C)^n$. If $E[(CT)^n]$ and $E[(CW)^n]$ are analytic at $\lambda = 0$, then so are $E[T^n]$ and $E[W^n]$.

Lemma 1. $T(\lambda)$ has finite moment generating function. Then according to Proposition 1, there exists a constant $C_T(\lambda) > 0$ (depending on λ) such that $E[T(\lambda)^n] \leq n!(C_T(\lambda))^n$. Furthermore, there is $\lambda_1 > 0$ such that $C_T(\lambda) < 1$ for $0 \leq \lambda \leq \lambda_1$.

Proof. The first part of lemma falls out directly from Thorisson [16]; also a different proof can be found in Wolff [18]. We now consider the second part of the lemma. From Weber [17], we know that $T(\lambda)$ is a.s. a convex function of λ. Also note $T(0) = 0$, therefore, $T(\lambda) \leq \lambda T(1)$ a.s. for $0 \leq \lambda \leq 1$. We choose λ_1 such that $\lambda_1 C_T(1) < 1$. Then, $E[T(\lambda)^n] \leq n!(\lambda_1 C_T(1))^n$ for $0 \leq \lambda \leq \lambda_1$. □

In the rest of this section, we will restrict ourselves to $\lambda \in [0, \lambda_1]$. It is well known that T and W satisfy

$$T = W + \lambda S \tag{15}$$

and

$$W \stackrel{d}{=} \max(T - Y, 0), \tag{16}$$

where $\stackrel{d}{=}$ means equal in distribution. We note W and S in (15) are independent of each other, as are T and Y in (16). Then it immediately follows from (15) that

$$\frac{E[T^n]}{n!} = \sum_{k=0}^{n} \frac{E[S^{n-k}]}{(n-k)!} \frac{E[W^k]}{k!} \lambda^{n-k}. \tag{17}$$

On the other hand, (16) gives

$$\begin{aligned} \frac{E[W^n]}{n!} &= E\left[\int_0^T \frac{(T-x)^n}{n!} dF(x)\right] \\ &= E\left[\int_0^T \frac{(T-x)^n}{n!} \sum_{k=0}^{\infty} \frac{F^{(k+1)}(0^+)}{k!} x^k dx + F(0^+) \frac{T^n}{n!}\right]. \end{aligned}$$

Since

$$\sum_{k=0}^{\infty}\int_0^T \left|\frac{(T-x)^n}{n!}\frac{F^{(k+1)}(0^+)}{k!}x^k\right| dx \;\le\; \sum_{k=0}^{\infty}\int_0^T \frac{(T-x)^n x^k}{n!k!}(C_F)^{k+1}dx = \sum_{k=0}^{\infty}\frac{T^{n+k+1}}{(n+k+1)!}(C_F)^{k+1} < \infty$$

and

$$\sum_{k=0}^{\infty}\frac{E[T^{n+k+1}]}{(n+k+1)!} \le \sum_{k=0}^{\infty}(C_T)^{n+k+1} < \infty$$

(note $C_F < 1$ and $C_T(\lambda) < 1$ for $0 \le \lambda \le \lambda_1$), by applying the dominated convergence theorem twice (one for interchanging the summation and the integration and the other for interchanging the summation and the expectation), we have

$$\frac{E[W^n]}{n!} = \sum_{k=0}^{\infty}F^{(k+1)}(0^+)\frac{E[T^{n+k+1}]}{(n+k+1)!} + F(0^+)\frac{E[T^n]}{n!} = \sum_{k=0}^{\infty}F^{(k)}(0^+)\frac{E[T^{n+k}]}{(n+k)!}. \tag{18}$$

Substituting (17) into (18), we obtain the following infinite system of linear equations for $\{E[W^n]/n!, n = 1, 2, \ldots\}$

$$\frac{E[W^n]}{n!} = \sum_{k=0}^{\infty}\alpha_k\sum_{i=0}^{n+k}\beta_{n+k-i}\frac{E[W^i]}{i!}\lambda^{n+k-i} = \sum_{k=0}^{\infty}\alpha_k\beta_{n+k}\lambda^{n+k} + \sum_{k=0}^{\infty}\alpha_k\sum_{i=1}^{n+k}\beta_{n+k-i}\frac{E[W^i]}{i!}\lambda^{n+k-i} \triangleq c_n + \sum_{j=1}^{\infty}c_{nj}\frac{E[W^j]}{j!}, \tag{19}$$

$n = 1, 2, \ldots$. (Recall $\alpha_i \triangleq F^{(i)}(0)$ and $\beta_i \triangleq E[S^i]/i!$.)

We first want to show that the infinite system of linear equations defined by (19) has a unique solution. We use the following result from Kantorovich and Krylov [10].

Lemma 2. Consider the infinite system of linear equations

$$\begin{bmatrix} 1-c_{11} & -c_{12} & -c_{13} & \cdots \\ -c_{21} & 1-c_{22} & -c_{23} & \cdots \\ -c_{31} & -c_{32} & 1-c_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix},$$

where c_n's and c_{nj}'s are real numbers satisfying

$$|c_n| \le M_1 \quad \text{and} \quad \sum_{j=1}^{\infty} |c_{nj}| \le M_2 < 1, \quad (n = 1, 2, \ldots)$$

where M_1 and M_2 are two constants. This system has one and only one bounded solution $\{z_n^*; n = 1, 2, \ldots\}$. Furthermore, if $\{z_n^N; n = 1, 2, \ldots, N\}$ is the solution of the finite system of linear equations

$$\begin{bmatrix} 1 - c_{11} & -c_{12} & \ldots & -c_{1N} \\ -c_{21} & 1 - c_{22} & \ldots & -c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{N1} & -c_{N2} & \ldots & 1 - c_{NN} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

then $\{z_n^N; n = 1, 2, \ldots, N\}$ is uniformly bounded, i.e., $|z_n^N| < M$ for some constant M, and

$$z_n^* = \lim_{N \to \infty} z_n^N, \quad (n = 1, 2, \ldots,).$$

Proof. See Kantorovich and Krylov [10] pp. 26-31. □

Lemma 3. There is $\lambda_2 > 0$ such that for every $\lambda \in [0, \lambda_2]$ the infinite system of linear equations defined by (19) has one and only one bounded solution.

Proof. To apply Lemma 2, we only need to verify

$$|c_n| \le M_1 \quad \text{and} \quad \sum_{j=1}^{\infty} |c_{nj}| \le M_2 < 1 \quad (n = 1, 2, \ldots).$$

Based on (14), we can choose $\lambda_2 \in (0, \lambda_1]$ such that

$$C_s\lambda_2 < 1 \quad \text{and} \quad \frac{1}{1 - C_s\lambda_2}\left(\frac{C_F}{1 - C_F} + \alpha_0\right) < 1.$$

Then, if $\lambda \in [0, \lambda_2]$, we first have

$$|c_n| \le \sum_{k=0}^{\infty} |\alpha_k| \beta_{n+k} \lambda^{n+k} \le \sum_{k=1}^{\infty} (C_F)^k (C_s\lambda_2)^{n+k} + \alpha_0 (C_s\lambda_2)^n \le \frac{C_F}{1 - C_F} + \alpha_0 < 1.$$

Secondly, we have

$$\begin{aligned} \sum_{j=1}^{\infty} |c_{nj}| &\le \sum_{k=0}^{\infty} |\alpha_k| \sum_{i=1}^{n+k} \beta_{n+k-i} \lambda^{n+k-i} \\ &\le \sum_{k=1}^{\infty} (C_F)^k \sum_{i=1}^{n+k} (C_s\lambda_2)^{n+k-i} + \alpha_0 \sum_{i=1}^{n} (C_s\lambda_2)^{n-i} \\ &\le \frac{1}{1 - C_s\lambda_2}\left(\frac{C_F}{1 - C_F} + \alpha_0\right) \\ &< 1. \end{aligned}$$

This completes the proof. □

Lemma 4. Suppose d_{nm}'s and b_{nm}'s are defined by recursive equations (3) and (4). We have

$$|d_{nm}| \le (C_1)^m \quad \text{and} \quad |b_{nm}| \le (C_1)^m \frac{C_1 - C_s}{C_s}, \tag{20}$$

for $m = 1, 3, \ldots$, and $n = 1, 2, \ldots, m$, where C_1 is any constant satisfying $C_1 > 2C_s$, and

$$\frac{C_1}{C_1 - C_s}\left(\frac{C_F}{1 - C_F} + \alpha_0\right) < 1.$$

Proof. Since $C_1 > 2C_s$, and

$$\frac{C_1}{C_1 - C_s}\left(\frac{C_F}{1 - C_F} + \alpha_0\right) < 1,$$

it is not difficult to show that

$$1 < \frac{C_1 - C_s}{C_s}, \tag{21}$$

and

$$\frac{1}{1 - \alpha_0}\left[\frac{C_F}{1 - C_F}\frac{C_1}{C_s} + \alpha_0\right] < \frac{C_1 - C_s}{C_s}. \tag{22}$$

We now use induction to prove (20). It is obvious that (20) holds for $m = 1$. Suppose (20) holds for $m \le k$. We now consider the case $m = k + 1$. For $n = k + 1$, we have

$$\begin{aligned}
|d_{k+1,k+1}| &\le |\beta_{k+1}| + \sum_{i=1}^{k} \beta_{k+1-i}|b_{ii}| \\
&\le (C_s)^{k+1} + \sum_{i=1}^{k} (C_s)^{k+1-i}(C_1)^i \frac{C_1 - C_s}{C_s} \\
&\le (C_1)^{k+1}\frac{C_1 - C_s}{C_s}\sum_{i=0}^{k}\left(\frac{C_s}{C_1}\right)^{k+1-i} \\
&\le (C_1)^{k+1}\frac{C_1 - C_s}{C_s}\frac{C_s}{C_1 - C_s} \\
&= (C_1)^{k+1},
\end{aligned}$$

and

$$|b_{k+1,k+1}| = \frac{\alpha_0}{1 - \alpha_0}|d_{k+1,k+1}| \le (C_1)^{k+1}\frac{C_1 - C_s}{C_s}.$$

Hence, (20) holds for $n = k + 1$. We assume that (20) holds for $n = l + 1, \ldots, k + 1$. Therefore, for $n = l$ we have

$$|d_{l(k+1)}| \le \sum_{i=1}^{l-1} |\beta_{l-i}||b_{i(k+1-l+i)}|$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{l-1}(C_s)^{l-i}(C_1)^{k+1-l+i}\frac{C_1-C_s}{C_s}\\
&= (C_1)^{k+1}\frac{C_1-C_s}{C_s}\sum_{i=1}^{l-1}(C_s/C_1)^{l-i}\\
&\leq (C_1)^{k+1}\frac{C_1-C_s}{C_s}\frac{C_s}{C_1-C_s}\\
&= (C_1)^{k+1},
\end{aligned}
$$

and

$$
\begin{aligned}
|b_{l(k+1)}| &\leq \frac{1}{1-\alpha_0}\left[\sum_{i=0}^{k-l}|\alpha_{i+1}|(|d_{(l+1+i)(k+1)}|+|b_{(l+1+i)(k+1)}|)+\alpha_0|d_{l(k+1)}|\right]\\
&\leq \frac{(C_F)^{k+1}}{1-\alpha_0}\left[\sum_{i=0}^{k-l}(C_F)^{i+1}\left(1+\frac{C_1-C_s}{C_s}\right)+\alpha_0\right]\\
&\leq (C_1)^{k+1}\frac{1}{1-\alpha_0}\left[\frac{C_F}{1-C_F}\frac{C_1}{C_s}+\alpha_0\right]\\
&\leq (C_1)^{k+1}\frac{C_1-C_s}{C_s}.
\end{aligned}
$$

(The last inequality is based on (22).) This completes the proof. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. From Lemma 4, we have

$$|a_{nm}| \leq (C_1)^m C_1/C_s \text{ and } |b_{nm}| \leq (C_1)^m (C_1-C_s)/C_s.$$

Hence there exists $\lambda_0 \in (0, \lambda_2]$ such that for any $\lambda \in [0, \lambda_0]$ the following two MacLaurin series

$$
\begin{aligned}
\tau_n(\lambda) &\triangleq \sum_{m=n}^{\infty} a_{nm}\lambda^m\\
\omega_n(\lambda) &\triangleq \sum_{m=n}^{\infty} b_{nm}\lambda^m
\end{aligned}
$$

($n = 1, 2, \ldots$) converge, and $\tau_n(\lambda) \leq C_1/C_s$ and $\omega_n(\lambda) \leq (C_1 - C_s)/C_s$. Since a_{nm}'s and b_{nm}'s are defined (through d_{nm}'s and b_{nm}'s) by recursive equations (3) and (4), it is not difficult to verify that for $0 \leq \lambda \leq \lambda_0$ $\{\omega_n(\lambda), n = 1, 2, \ldots\}$ satisfies

$$\omega_n(\lambda) = \sum_{k=0}^{\infty}\alpha_k\beta_{n+k}\lambda^{n+k} + \sum_{k=0}^{\infty}\alpha_k\sum_{i=1}^{n+k}\beta_{n+k-i}\omega_i(\lambda)\lambda^{n+k-i}.$$

We can also verify

$$\tau_n(\lambda) = \sum_{k=0}^{n} \beta_{n-k}\omega_k(\lambda)\lambda^{n-k}.$$

Therefore, both $\{E[W(\lambda)^n]/n!, n = 1, 2, \ldots\}$ and $\{\omega_n(\lambda), n = 1, 2, \ldots\}$ are bounded solutions to the infinite system of linear equations defined by (19), which according to Lemma 3 has only one bounded solution. This immediately leads to

$$E[W(\lambda)^n]/n! = \sum_{m=n}^{\infty} b_{nm}\lambda^m, \quad \text{for } 0 \le \lambda \le \lambda_0.$$

And we also have

$$E[T(\lambda)^n]/n! = \sum_{m=n}^{\infty} a_{nm}\lambda^m, \quad \text{for } 0 \le \lambda \le \lambda_0.$$

This completes the proof. □

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